

Elliptic Curves in Moduli Space of Stable Bundles of Rank 3

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Abstract

Let M be the moduli space of rank 3 stable bundles with fixed determinant of degree 1 on a smooth projective curve of genus $g \geq 2$. When C is generic, we show that any essential elliptic curve on M has degree (respect to anti-canonical divisor $-K_M$) at least 6, and we give a complete classification for elliptic curves of degree 6, which is not same as Sun's Conjecture. Moreover, if $g > 12$, we show that any elliptic curve passing through the generic point of M has degree at least 18.

1 Introduction

Let C be a smooth projective curve of genus $g \geq 2$ and \mathcal{L} be a line bundle on C of degree d . Let $M := SU_C(r, \mathcal{L})$ be the moduli space of stable vector bundles of rank r and with the fixed determinant \mathcal{L} , which is a smooth quasi-projective Fano-variety with $Pic(M) = \mathbb{Z} \cdot \Theta$. And $-K_M = 2(r, d)\Theta$, where Θ is an ample divisor ([1] [2]). Let B be a smooth projective curve of genus b . The degree of a curve $\phi : B \rightarrow M$ is defined to be $\deg \phi^*(-K_M)$. It seems quite natural to ask what is the lower bound of degree and to classify the curves of lower degree.

When $b = 0$, i.e., $B = \mathbb{P}^1$, it has been proved that any rational curve $\phi : \mathbb{P}^1 \rightarrow M$ passing through the generic point has degree at least $2r$ provided that $(r, d) = 1$. Moreover, it has degree $2r$ if and only if it is a Hecke curve ([4], Theorem 1). Ramanan [1] found a family of lines on M , i.e., rational curves $\phi : \mathbb{P}^1 \rightarrow M$ such that $\deg \phi^*(-K_M) = 2(r, d)$. And all the lines are determined in [4] and [3]. In [5], we have studied the small rational curves (i.e., the rational curves have degrees smaller than $2r$) on M and estimate the codimension of the locus of the small rational curves when $d = 1$; in particular, we determinant all small rational curves when $r = 3$ ([5]). Thus it is natural to ask what are the situation when $b > 0$.

When $b = 1$, it may happen that the normalization of $\phi(B)$ is \mathbb{P}^1 . To avoid this case, we only consider the case that $\phi : B \rightarrow M$ is an essential elliptic curve (cf. [6]). The paper [6] is a start to study the case of $b = 1$. In [6], Sun constructed essential elliptic curves of degree $6(r, d)$ on M , which are called elliptic curves of split type, and essential elliptic curves of degree $6r$ that passing through the generic point of M , which are called elliptic curves of Hecke type. Do they exhaust all minimal essential elliptic curves on M (resp. minimal essential elliptic curves passing through generic point of M)? For this, Sun gave the following conjecture (cf. Conjecture 4.8 of [6]):

Sun's Conjecture: *Let $\phi : B \rightarrow M = SU_C(r, \mathcal{L})$ is an essential elliptic curve defined by a vector bundle E on $C \times B$. Then, when C is a generic curve, we have*

$$\deg \phi^*(-K_M) = \Delta(E) \geq 6(r, d)$$

and $\deg \phi^(-K_M) = 6(r, d)$ if and only if it is an elliptic curve of split type with minimal degree. If $\phi : B \rightarrow M$ passes through the generic point and $g > 4$, then $\deg \phi^*(-K_M) \geq 6r$.*

In [6], Sun also gave an positive answer to this conjecture when $r = 2$ and $d = 1$. In this paper, we consider the case that $r = 3$ and $d = 1$, then M is a smooth projective Fano-variety of dimension $8g - 8$. When C is generic, we show that any essential elliptic curve $\phi : B \rightarrow M$ has degree at least 6 (see Theorem 4.12). When $g > 12$ and C is generic, we show that any essential elliptic curve $\phi : B \rightarrow M$ passing through the generic point of M have degree at least 18 (see Theorem 4.14). But an essential elliptic curve of degree 6 may not be an elliptic curve of split type (cf. Proposition 3.6 and Theorem 4.12).

We give a brief description of the article. In section 2, we recall a degree formula of curves for general case which has proven in [6]. In section 3, we recall the constructions of **elliptic curves of Hecke type** and **elliptic curves of split type**. And, we also give a class of degree 6 essential elliptic curves which are not elliptic curves of split type when $r = 3$ and $d = 1$. In section 4, we

prove the main theorems (Theorem 4.12 and Theorem 4.14), which partly prove Sun's conjecture for the case $r = 3, d = 1$. On the other hand, Theorem 4.12 also implies that the essential elliptic curves of degree 6 may not be elliptic curves of split type.

2 The degree formula of curves in moduli spaces

Let's recall the degree formula of curves in moduli spaces.

Lemma 2.1. ([6]) *For any smooth projective curve B of genus b , if $\phi : B \rightarrow M$ is defined by a vector bundle E of rank r on $C \times B$. Then*

$$\deg \phi^*(-K_M) = c_2(\mathcal{E}nd^0(E)) = 2rc_2(E) - (r-1)c_1(E)^2 := \Delta(E).$$

Let $f : X := C \times B \rightarrow C$ be the projection. Then for any vector bundle E on X , there is a relative Harder-Narasimhan filtration (cf. Theorem 2.3.2, page 45 in [7])

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that $F_i = E_i/E_{i-1}$ ($i = 1, \dots, n$) are flat over C and its restriction to general fiber $X_t = f^{-1}(t)$ is the Harder-Narasimhan filtration of $E|_{X_t}$. Thus F_i are semi-stable of slope μ_i at generic fiber of $f : X \rightarrow C$ with $\mu_1 > \mu_2 > \cdots > \mu_n$. Then we have

Theorem 2.2. ([6]) *For any vector bundle of rank r on X , let*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

be the relative Harder-Narasimhan filtration over C with $F_i = E_i/E_{i-1}$ and $\mu_i = \mu(F_i|_{f^{-1}(x)})$ for generic $x \in C$. Let $\mu(E)$ and $\mu(E_i)$ denote the slope of $E|_{\pi^{-1}(b)}$ and $E_i|_{\pi^{-1}(b)}$ for generic $b \in B$. Then, if

$$\text{Pic}(C \times B) = \text{Pic}(C) \times \text{Pic}(B),$$

we have the following formula

$$\Delta(E) = 2r \left(\sum_{i=1}^n (c_2(F_i) - \frac{rk(F_i)-1}{2rk(F_i)} c_1(F_i)^2) + \sum_{i=1}^{n-1} (\mu(E) - \mu(E_i)) rk(E_i) (\mu_i - \mu_{i+1}) \right). \quad (1)$$

Remark 2.3. (i) *The assumption $\text{Pic}(C \times B) = \text{Pic}(C) \times \text{Pic}(B)$ is always hold when $B = \mathbb{P}^1$;*
(ii) *The assumption also holds when B is an elliptic curve and C is generic.*

Theorem 2.4. ([6]) *For any torsion free sheaf \mathcal{F} on $X = C \times B$, if its restriction to a fiber of $f : X \rightarrow C$ is semi-stable, then*

$$\Delta(\mathcal{F}) = 2rk(\mathcal{F})c_2(\mathcal{F}) - (rk(\mathcal{F}) - 1)c_1(\mathcal{F})^2 \geq 0.$$

*If the determinants $\{\det(\mathcal{F}^{**})_x\}_{x \in C}$ are isomorphic each other, then $\Delta(\mathcal{F}) = 0$ if and only if \mathcal{F} is locally free and satisfies*

- *All the bundles $\{\mathcal{F}_x := \mathcal{F}|_{\{x\} \times B}\}_{x \in C}$ are semi-stable and s-equivalent each other.*
- *All the bundles $\{\mathcal{F}_y := \mathcal{F}|_{C \times \{y\}}\}_{y \in B}$ are isomorphic each other.*

We will need the following lemma in the later computation, whose proof are straightforward computations. Recall that $X_t = f^{-1}(t)$ denotes the fiber of $f : X \rightarrow C$ and for any vector bundle \mathcal{F} on X , \mathcal{F}_t denote the restrictions of \mathcal{F} to X_t .

Lemma 2.5. ([6]) *Let $\mathcal{F}_t \rightarrow W \rightarrow 0$ be a locally free quotient and*

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow_{X_t} W \rightarrow 0$$

be the elementary transformation of \mathcal{F} along W at $X_t \subset X$. Then

$$\Delta(\mathcal{F}) = \Delta(\mathcal{F}') + 2r(\mu(\mathcal{F}_t) - \mu(W))rkW.$$

3 Examples of Elliptic Curves on Moduli Spaces and Sun's Conjecture

Recall that given two nonnegative integers k, l , a vector bundle W of rank r and degree d on C is (k, l) -stable, if, for each proper subbundle W' of W , we have

$$\frac{\deg(W') + k}{\text{rk}(W')} < \frac{\deg(W) + k - l}{r}.$$

Remark 3.1. (i) *The usual stability is equivalent to $(0, 0)$ -stability.*

(ii) *If W is (k, l) -stable, then W^* is (l, k) -stable.*

(iii) *The (k, l) -stability is an open condition.*

Let $M := SU_C(r, \mathcal{L})$ be the moduli space of stable vector bundles of rank r and with the fixed determinant \mathcal{L} and $\deg \mathcal{L} = d$. If (k, l) -stable points exist, then the set of (k, l) -stable points is open in M . And the (k, l) -stable points are the so called generic points. So it's natural to ask that does the (k, l) -stable point exist?

It's equivalent to estimate the dimension of the subvariety of $M := SU_C(r, \mathcal{L})$ consisting of non- (k, l) -stable points. Clearly any such bundle E contains a subbundle F satisfying the inequality

$$\frac{\deg F + k}{\text{rk} F} \geq \frac{\deg E + k - l}{\text{rk} E} = \frac{d + k - l}{r}.$$

By using [proposition 2.6 of [8]] as in [Lemma 6.7 of [8]], we may as well assume that F and E/F are stable and compute the dimension of such bundles E . The dimension of a component corresponding to a fixed rank n and degree δ of F such that $\frac{\delta+k}{n} \geq \frac{d+k-l}{r}$ is majored by $\dim U(n, \delta) + \dim U(r-n, d-\delta) + \dim H^1(C, \text{Hom}(E/F, F)) - 1 - g = (r^{\frac{\delta}{2}} - 1)(g-1) - n(r-n)(g-1) + (nd-r\delta)$. If all the dimensions of the components is strictly smaller than the dimension of M , then (k, l) -stable point exists. Thus we hope that $-n(r-n)(g-1) + (nd-r\delta) < 0$ for any n and δ such that $\frac{\delta+k}{n} \geq \frac{d+k-l}{r}$, it's necessary to prove that

$$(r-n)k + nl < n(r-n)(g-1) \text{ for any } 1 \leq n \leq r-1. \quad (2)$$

Thus we have following lemmas, which proof are easy and elementary (cf. [9]).

Lemma 3.2. *If $g \geq 3$, M contains $(0, 1)$ -stable and $(1, 0)$ -stable bundles. M contains a $(1, 1)$ -stable bundle W except $g = 3, d, r$ both even.*

Lemma 3.3. *Let $0 \rightarrow V \rightarrow W \rightarrow \mathcal{O}_p \rightarrow 0$ be an exact sequence, where \mathcal{O}_p is the 1-dimensional skyscraper sheaf at $p \in C$. If W is (k, l) -stable, then V is $(k, l-1)$ -stable.*

At first, let's recall a class of elliptic curves passing through a generic point, which are called **elliptic curves of Hecke type**. Let $U_C(r, d-1)$ be the moduli space of stable bundles of rank r and degree $d-1$. Let $\mathfrak{D} \subset U_C(r, d-1)$ be the open set of $(1, 0)$ -stable bundles. Let $\psi : C \times \mathfrak{D} \rightarrow J^d(C)$ be defined as $\psi(x, V) = \mathcal{O}_C(x) \otimes \det(V)$ and $\mathfrak{R}_C := \psi^{-1}(\mathcal{L}) \subset C \times \mathfrak{D}$ be the fibre of ψ at the point $[\mathcal{L}] \in J^d(C)$. There exists a projective bundle

$$p : \mathfrak{P} \rightarrow \mathfrak{R}_C$$

such that for any $(x, V) \in \mathfrak{R}_C$ we have $p^{-1}(x, V) = \mathbb{P}(V_x^*)$. Let $V_x^* \otimes \mathcal{O}_{\mathbb{P}(V_x^*)} \rightarrow \mathcal{O}_{\mathbb{P}(V_x^*)}(1) \rightarrow 0$ be the universal quotient, $f : C \times \mathbb{P}(V_x^*) \rightarrow C$ be the projection, and

$$0 \rightarrow \mathfrak{E}^* \rightarrow f^* V^* \rightarrow_{\{x\} \times \mathbb{P}(V_x^*)} \mathcal{O}_{\mathbb{P}(V_x^*)}(1) \rightarrow 0$$

where \mathfrak{E}^* is defined to the kernel of the surjection. Take dual, we have

$$0 \rightarrow f^* V \rightarrow \mathfrak{E} \rightarrow_{\{x\} \times \mathbb{P}(V_x^*)} \mathcal{O}_{\mathbb{P}(V_x^*)}(-1) \rightarrow 0, \quad (3)$$

which, at any point $\xi = (V_x^* \rightarrow \Lambda \rightarrow 0) \in \mathbb{P}(V^*)$, gives exact sequence

$$0 \longrightarrow V \xrightarrow{\iota} \mathfrak{E}_\xi \longrightarrow \mathcal{O}_x \rightarrow 0$$

on C such that $\ker(\iota_x) = \Lambda^* \subset V_x$. V being $(1,0)$ -stable implies stability of \mathfrak{E}_ξ . Thus (3) defines

$$\Psi_{(x,V)} : \mathbb{P}(V_x^*) = p^{-1}(x, V) \rightarrow M. \quad (4)$$

Definition 3.4. (cf. Definition 3.4 of [6]) The images (under $\{\Psi_{(x,V)}\}_{(x,V) \in \mathfrak{R}_C}$) of lines in the fiber of $p : \mathfrak{P} \rightarrow \mathfrak{R}_C$ are the so called **Hecke curves** in M . The images (under $\{\Psi_{(x,V)}\}_{(x,V) \in \mathfrak{R}_C}$) of elliptic curves in the fibers of $p : \mathfrak{P} \rightarrow \mathfrak{R}_C$ are called **elliptic curves of Hecke type**.

It's known that (cf. Lemma 5.9 of [9]) that the morphisms in (4) are closed immersion, and the images of smooth elliptic curves $B \subset \mathbb{P}(V_x^*)$ with degree 3 are smooth elliptic curves on M that pass through generic point of M , which are elliptic curves of Hecke type and have degree $6r$ (cf. Example 3.5 of [6]).

If we do not require the curve $\phi : B \rightarrow M$ passing through generic point of M , there are elliptic curves with smaller degree. Now, let's recall a class elliptic curves passing through generic points of M , which are called *elliptic curves of split type*. For any given r and d , let r_1, r_2 be positive integers and d_1, d_2 be integers that satisfy the equalities $r_1 + r_2 = r, d_1 + d_2 = d$ and

$$r_1 \frac{d}{(r, d)} - d_1 \frac{r}{(r, d)} = 1, d_2 \frac{r}{(r, d)} - r_2 \frac{d}{(r, d)} = 1.$$

Let $U_C(r_1, d_1)$ (resp. $U_C(r_2, d_2)$) be the moduli space of stable vector bundles with rank r_1 (resp. r_2) and degree d_1 (resp. d_2). Then, since $(r_1, d_1) = 1$ and $(r_2, d_2) = 1$, there is are universal bundles $\mathcal{V}_1, \mathcal{V}_2$ on $C \times U_C(r_1, d_1)$ and $C \times U_C(r_2, d_2)$ respectively. Consider

$$U_C(r_1, d_1) \times U_C(r_2, d_2) \xrightarrow{\det(\bullet) \times \det(\bullet)} J_C^{d_1} \times J_C^{d_2} \xrightarrow{(\bullet) \otimes (\bullet)} J_C^d,$$

let $\mathcal{R}(r_1, d_1)$ be its fiber at $[\mathcal{L}] \in J_C^d$. The pullback of $\mathcal{V}_1, \mathcal{V}_2$ by the projection $C \times \mathcal{R}(r_1, d_1) \rightarrow C \times U_C(r_i, d_i) (i = 1, 2)$ is still denoted by $\mathcal{V}_1, \mathcal{V}_2$ respectively. Let $p : C \times \mathcal{R}(r_1, d_1) \rightarrow \mathcal{R}(r_1, d_1)$ and $\mathcal{G} = R^1 p_*(\mathcal{V}_2^* \otimes \mathcal{V}_1)$, which is locally free of rank $r_1 r_2 (g - 1) + (r, d)$. Let

$$q : P(r_1, d_1) = \mathbb{P}(\mathcal{G}) \rightarrow \mathcal{R}(r_1, d_1)$$

be the projective bundle parametrizing 1-dimensional subspaces of $\mathcal{G}_t (t \in \mathcal{R}(r_1, d_1))$ and $f : C \times P(r_1, d_1) \rightarrow C, \pi : C \times P(r_1, d_1) \rightarrow P(r_1, d_1)$ be the projections. Then there is a universal extension

$$0 \rightarrow (id \times q)^* \mathcal{V}_1 \otimes \pi^* \mathcal{O}_{P(r_1, d_1)}(1) \rightarrow \mathcal{E} \rightarrow (id \times q)^* \mathcal{V}_2 \rightarrow 0 \quad (5)$$

on $C \times P(r_1, d_1)$ such that for any $x = ([V_1], [V_2], [e]) \in P(r_1, d_1)$, where $[V_i] \in U_C(r_i, d_i)$ with $\det(V_1)\det(V_2) = \mathcal{L}$ and $[e] \subset H^1(C, V_2^* \otimes V_1)$ being a line through a origin, the bundle $\mathcal{E}|_{C \times \{x\}}$ is the isomorphic class of vector bundles V given by extensions

$$0 \rightarrow V_1 \rightarrow V \rightarrow V_2 \rightarrow 0$$

that defined by vectors on the line $[e] \subset H^1(C, V_2^* \otimes V_1)$. Then V must be stable (cf. Lemma 2.2 of [3]), and the sequence (5) defines

$$\Phi : P(r_1, d_1) \rightarrow SU_C(r, \mathcal{L}) = M.$$

On each fiber $q^{-1}(\xi) = \mathbb{P}(H^1(V_2^* \otimes V_1))$ at $\xi = (V_1, V_2)$, the morphisms

$$\Phi_\xi := \Phi|_{q^{-1}(\xi)} : q^{-1}(\xi) = \mathbb{P}(H^1(V_2^* \otimes V_1)) \rightarrow M \quad (6)$$

is birational and $\Phi_\xi^*(-K_M) = \mathcal{O}_{\mathbb{P}(H^1(V_2^* \otimes V_1))}(2(r, d))$ (cf. Lemma 2.4 of [3]).

Definition 3.5. (cf. Example 3.6 of [6]) The images (under $\{\Phi_\xi\}_{\xi \in \mathcal{R}(r_1, d_1)}$) of smooth elliptic curves in the fibers of $q : P(r_1, d_1) = \mathbb{P}(\mathcal{G}) \rightarrow \mathcal{R}(r_1, d_1)$ are called **elliptic curves of split type**. For any smooth elliptic curve $B \subset q^{-1}(\xi) = \mathbb{P}(H^1(V_2^* \otimes V_1))$ of degree 3, the image of $\Phi_\xi|_B : B \rightarrow M$ is of degree $6(r, d)$, which is so called **elliptic curves of split type with minimal degree**.

When $r = 2$ and $d = 1$, Sun has shown that any essential elliptic curve has degree at least 6, it has degree 6 if and only if it is an elliptic curve of split type with minimal degree. And then Sun conjectures the result holds for any rank r and degree d (i.e., **Sun's conjecture**). But when $r = 3, d = 1$, there is a class of elliptic curves in M of degree 6, which are not the elliptic curves of split type.

Note that when $r = 3, d = 1$, we must have $r_1 = 1, d_1 = 0$ and let $\mathcal{R}_\mathcal{L} := \mathcal{R}(0, 0), \mathcal{P} := P(0, 0)$.

Proposition 3.6. Let $B \subset \mathcal{P}$ be an elliptic curve, which is mapped to a point in J_C but is not in any fiber of q and $\deg(\mathcal{O}_\mathcal{P}(1)|_B) = 1$. If the normalization of the image of B induced by q is a line in $SU_C(2, \mathcal{L}')$ for some degree 1 line bundle \mathcal{L}' on C and B is a degree 2 cover over its image in $SU_C(2, \mathcal{L}')$, then $\Phi|_B : B \rightarrow M$ is an essential elliptic curve on M of degree 6.

Proof. Let $p_1 : \mathcal{R}_\mathcal{L} \rightarrow J_C, p_2 : \mathcal{R}_\mathcal{L} \rightarrow U_C(2, 1)$ be the projections. If the normalization of the image of B induced by q is a line in $SU_C(2, \mathcal{L}')$, then $p_1 \circ q(B) = [\mathcal{L} \otimes \mathcal{L}'^{-1}]$ is a point of J_C and let $L_1 := \mathcal{L} \otimes \mathcal{L}'^{-1}$. And then by the results of lines in [4] and [3], there are line bundles L_2 and L_3 on C of degrees 0 and 1 respectively with $L_2 \otimes L_3 = \mathcal{L}'$, such that $B \rightarrow SU_C(2, \mathcal{L}')$ factors as the composition of $\varphi : B \rightarrow \mathbb{P}H^1(L_3^{-1} \otimes L_2)$ with $\theta : \mathbb{P}H^1(L_3^{-1} \otimes L_2) \rightarrow SU_C(2, \mathcal{L}')$ such that $\varphi^* \mathcal{O}_{\mathbb{P}H^1(L_3^{-1} \otimes L_2)} = \mathcal{O}_B(2)$. Where $\theta : \mathbb{P}H^1(L_3^{-1} \otimes L_2) \rightarrow SU_C(2, \mathcal{L} \otimes L_1^{-1})$ is a closed immersion defined by a vector bundle \mathcal{E}' on $C \times \mathbb{P}H^1(L_3^{-1} \otimes L_2)$ satisfying an exact sequence

$$0 \rightarrow f^* L_2 \otimes \pi^* \mathcal{O}_{\mathbb{P}H^1(L_3^{-1} \otimes L_2)}(1) \rightarrow \mathcal{E}' \rightarrow f^* L_3 \rightarrow 0. \quad (7)$$

By the construction of $q : \mathcal{P} \rightarrow \mathcal{R}_\mathcal{L}$, the restriction of (5) to $C \times B$ equals to

$$0 \rightarrow f^* L_1 \otimes \pi^* \mathcal{O}(1) \rightarrow E \rightarrow E' \rightarrow 0, \quad (8)$$

where $E' := (id_C \times \varphi)^* \mathcal{E}'$ satisfying

$$(0 \rightarrow f^* L_2 \otimes \pi^* \mathcal{O}_B(2) \rightarrow E' \rightarrow f^* L_3 \rightarrow 0) \cong (id_C \times \varphi)^*(7). \quad (9)$$

Since Φ is defined by the sequence (5), so $\Phi|_B : B \rightarrow M$ is defined by the sequence (8). Thus $\deg \Phi|_B^*(-K_M) = \Delta(E)$. By considering sequences (8) and (9), we have $\Delta(E) = 6$. \square

Remark 3.7. In fact, a elliptic curve in above proposition is defined by a vector bundle E on $C \times B$ fits following exact sequences

$$0 \rightarrow f^* L_1 \otimes \pi^* \mathcal{O}(1) \longrightarrow E \longrightarrow E' \rightarrow 0,$$

$$0 \rightarrow f^* L_2 \otimes \pi^* \mathcal{O}(2) \longrightarrow E' \longrightarrow f^* L_3 \rightarrow 0,$$

where L_1, L_2 are degree 0 line bundle on C and L_3 is a degree 1 line bundle on C .

4 Minimal Elliptic Curves on Moduli Spaces

In this section, we consider the moduli space M of rank 3 stable bundles on C with a fixed determinant \mathcal{L} of degree 1. We also assume that the curve C is generic in the sense that C admits no surjective morphism to an elliptic curve. With this assumption, we know that $\text{Pic}(C \times B) = \text{Pic}(C) \times \text{Pic}(B)$ for any elliptic curve B .

For a morphism $\phi : B \rightarrow M$, it may happen that the normalization of $\phi(B)$ is a rational curve. To avoid this case, we assume that $\phi : B \rightarrow M$ is an essential elliptic curve of M in this section.

Let E be the vector bundle on $X = C \times B$ that defines ϕ . Consider the relative Harder-Narasimhan filtration (cf Theorem 2.3.2, page 45 in [7])

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

over C . When $r = 3$, there are three choices for $n : 1, 2$ and 3 .

Proposition 4.1. *When $n = 3$, we have $\Delta(E) \geq 10$. If $g \geq 3$ and $\phi : B \rightarrow M$ passes through a generic point of M , then $\Delta(E) \geq 18$.*

Proof. Let $0 = E_0 \subset E_1 \subset E_2 \subset E_3 = E$ be the relative Harder-Narasimhan filtration over C . Let $F_i = E_i/E_{i-1}$ ($i = 1, 2, 3$), then we have exact sequences

$$0 \rightarrow E_1|_{X_t} \rightarrow E_2|_{X_t} \rightarrow F_2|_{X_t} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow E_2|_{X_t} \rightarrow E|_{X_t} \rightarrow F_3|_{X_t} \rightarrow 0$$

on each fiber $X_t = \{t\} \times B$ of $f : X \rightarrow C$ since $\{F_i\}_{i=1,2,3}$ are flat over C . Thus E_1 is locally free (cf Lemma 1.27 of [10]) and by the Theorem 2.2

$$\Delta(E) = 6c_2(F_2) + 6c_2(F_3) + (2 - 6\deg(E_1))(\mu_1 - \mu_2) + (4 - 6\deg(E_2))(\mu_2 - \mu_3). \quad (10)$$

where $\mu_i = \mu(F_i|_{X_t})$ ($i = 1, 2, 3$) for generic $t \in C$.

Note that $c_2(F_i) \geq 0$ ($i = 2, 3$) since F_i ($i = 2, 3$) are semi-stable on generic fiber of $f : X \rightarrow C$. Let $d_i = \deg(E_i) - \deg(E_{i-1})$, then $d_1 \leq 0$, $d_1 + d_2 \leq 0$ and $d_1 + d_2 + d_3 = 1$ since $E_y = E|_{C \times \{y\}}$ is stable of degree 1 for any $y \in B$.

If $\exists c_2(F_i) \neq 0$ ($i = 2$ or 3), then $\Delta(E) = 6c_2(F_2) + 6c_2(F_3) + 2(\mu_1 - \mu_2) + 4(\mu_2 - \mu_3) \geq 12$. If $\phi : B \rightarrow M$ passes through a generic point, i.e., a $(1,1)$ -stable point, which implies $\deg(E_1) \leq -1$ and $\deg(E_2) \leq -1$. Thus

$$\Delta(E) \geq 6 + 8(\mu_1 - \mu_2) + 10(\mu_2 - \mu_3) \geq 24.$$

From now, we will assume that $c_2(F_2) = c_2(F_3) = 0$. If $d_1 \neq 0$, we must have $\Delta(E) \geq (2 - 6(-1))(\mu_1 - \mu_2) + (4 - 6\deg(E_2))(\mu_2 - \mu_3) \geq 12$. And, if $\phi : B \rightarrow M$ passes through a generic point, then

$$\Delta(E) \geq (2 - 6(-1))(\mu_1 - \mu_2) + (4 - 6(-1))(\mu_2 - \mu_3) \geq 18.$$

There left one case we need to consider when $c_2(F_2) = c_2(F_3) = 0$ and $d_1 = 0$. In this case, we note that $\phi : B \rightarrow M$ can not pass through any generic point of M , F_2 and F_3 are line bundles and there are line bundles $\mathcal{L}_1, \mathcal{L}_2$ and \mathcal{L}_3 on C of degrees $0, d_2$ and d_3 respectively, such that

$$E_1 = f^*\mathcal{L}_1 \otimes \pi^*\mathcal{O}(\mu_1), \quad F_2 = f^*\mathcal{L}_2 \otimes \pi^*\mathcal{O}(\mu_2) \quad \text{and} \quad F_3 = f^*\mathcal{L}_3 \otimes \pi^*\mathcal{O}(\mu_3)$$

where $\mathcal{O}(\mu_i)$ denote a line bundle of degree μ_i on B ($i = 1, 2, 3$). Replace E by $E \otimes \pi^*\mathcal{O}(-\mu_3)$, we can assume that $\mu_3 = 0$ and $\mu_1 > \mu_2 > 0$. Now we have exact sequences

$$0 \rightarrow f^*\mathcal{L}_1 \otimes \pi^*\mathcal{O}(\mu_1) \rightarrow E_2 \rightarrow f^*\mathcal{L}_2 \otimes \pi^*\mathcal{O}(\mu_2) \rightarrow 0$$

and

$$0 \rightarrow E_2 \rightarrow E \rightarrow f^*\mathcal{L}_3 \rightarrow 0.$$

Let $E' := E/(f^*\mathcal{L}_1 \otimes \pi^*\mathcal{O}(\mu_1))$, then there is an induced morphism $\alpha : E' \rightarrow f^*\mathcal{L}_3$ satisfying the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & f^*\mathcal{L}_1 \otimes \pi^*\mathcal{O}(\mu_1) & \longrightarrow & E & \longrightarrow & E' \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \alpha \\ 0 & \longrightarrow & E_2 & \longrightarrow & E & \longrightarrow & f^*\mathcal{L}_3 \longrightarrow 0. \end{array}$$

By the Snake Lemma, α is surjective and $\ker(\alpha) \cong f^*\mathcal{L}_2 \otimes \pi^*\mathcal{O}(\mu_2)$. Then E' fits an exact sequence

$$0 \rightarrow f^*\mathcal{L}_2 \otimes \pi^*\mathcal{O}(\mu_2) \rightarrow E' \rightarrow f^*\mathcal{L}_3 \rightarrow 0$$

which induces a morphism

$$\psi : B \longrightarrow \mathbb{P}H^1(\mathcal{L}_3^{-1} \otimes \mathcal{L}_2) = \mathbb{P}^{g-2+d_3-d_2}$$

such that $\psi^* \mathcal{O}_{\mathbb{P}^{g-2+d_3-d_2}}(1) = \mathcal{O}(\mu_2)$. Thus $\mu_2 \geq 2$ and

$$\Delta(E) = 2(\mu_1 - \mu_2) + 4(\mu_2 - \mu_3) \geq 2 + 4 \times 2 = 10.$$

□

When $n = 2$, let $0 \rightarrow E_1 \rightarrow E \rightarrow F_2 \rightarrow 0$ be the relative Harder-Narasimhan filtration over C , then we have an exact sequence

$$0 \rightarrow E_1|_{X_t} \rightarrow E|_{X_t} \rightarrow F_2|_{X_t} \rightarrow 0$$

on each fiber $X_t = f^{-1}(t)$ of $f : X \rightarrow C$ since E_1, F_2 are flat over C . Thus E_1 is locally free (cf. Lemma 1.27 of [10]) and F_2 is locally free over $f^{-1}(C \setminus T)$ where $T \subset C$ is a finite set. Thus

$$0 \rightarrow E_1|_{C \times \{y\}} \rightarrow E|_{C \times \{y\}} \rightarrow F_2|_{C \times \{y\}} \rightarrow 0, \text{ for any } y \in B \quad (11)$$

are exact sequences, which imply that F_2 is also B -flat.

When $n = 2$, there are two cases: $\text{rk} E_1 = 2$ and $\text{rk} E_1 = 1$.

Lemma 4.2. *If $g \geq 4$, M contains $(2, 0)$ -stable points.*

Proof. For $1 \leq n \leq 3 - 1 = 2$ and $(k, l) = (2, 0)$, the inequality (2) always holds when $g \geq 4$. Thus M contains $(2, 0)$ -stable points when $g \geq 4$. □

Proposition 4.3. *When $n = 2$ and $\text{rk} E_1 = 2$, $\Delta(E) \geq 6$. If $g \geq 4$ and $\phi : B \rightarrow M$ passes through the generic points, $\Delta(E) \geq 20$.*

Proof. In this case, tensoring E with $\pi^* L$ for some suitable line bundle L on B , we can assume that $\mu_1 = 0$ or $\frac{1}{2}$, $\mu_1 > \mu_2$ and

$$\Delta(E) = \frac{3}{2}\Delta(E_1) + 6c_2(F_2) + (4 - 6\deg E_1)(\mu_1 - \mu_2). \quad (12)$$

If $\Delta(E_1) \neq 0$, then $\Delta(E_1) = 4c_2(E_1) - 2d_1(2\mu_1) \geq 2$ by Theorem 2.4 and the fact that $\text{Pic}(C \times B) = \text{Pic}(C) \times \text{Pic}(B)$. If $d_1 = 0$, then $\Delta(E_1) = 4c_2(E_1) \in 4\mathbb{Z}$ and $\Delta(E) \geq \frac{3}{2} \cdot 4 + 4 \cdot \frac{1}{2} = 8$. If $d_1 \leq -1$, then $\Delta(E) \geq \frac{3}{2} \cdot 2 + 10 \cdot \frac{1}{2} = 8$.

When $\phi : B \rightarrow M$ passes through a generic point, i.e., a $(2, 0)$ -stable point, which implies $d_1 = \deg E_1 \leq -1$.

If $\mu_1 = 0$, then $\Delta(E_1) = 4c_2(E_1) - 4d_1\mu_1 \geq 4$. When $\deg E_1 \leq -2$, it's easy to see that $\Delta(E) \geq \frac{3}{2} \cdot 4 + (4 - 6 \cdot (-2)) = 22$. Now we assume that $\deg E_1 = -1$. E_{1y} is stable of degree -1 for generic $y \in B$ since E_y is $(2, 0)$ -stable. And we can prove that $\Delta(E_1) \geq 8$ (same as the proof of Proposition 4.5 in [6]), thus $\Delta(E) \geq \frac{3}{2} \cdot 8 + (4 - 6 \cdot (-1)) = 22$.

If $\mu_1 = \frac{1}{2}$, which means that E_1 is semi-stable of degree 1 at the generic fiber of $f : X \rightarrow C$. It's known that there is a unique stable rank 2 vector bundle with a fixed determinant of degree 1 on an elliptic curve. Thus $\Delta(E_1) > 0$ if and only if there exists $t_1 \in C$ such that $E_{1t_1} = E_1|_{X_{t_1}}$ is not semi-stable. Let $E_{1t_1} \rightarrow \mathcal{O}(\mu_{1,1}) \rightarrow 0$ be the quotient of minimal degree and

$$0 \rightarrow E_1^{(1)} \rightarrow E_1 \rightarrow_{X_{t_1}} \mathcal{O}(\mu_{1,1}) \rightarrow 0$$

be the elementary transform of E_1 along $\mathcal{O}(\mu_{1,1})$ at X_{t_1} . If $E_1^{(i)}$ is defined and $\Delta(E_1^{(i)}) > 0$, let $t_{i+1} \in C$ such that $E_{1t_{i+1}}^{(i)} = E_1^{(i)}|_{X_{t_{i+1}}}$ is not semi-stable and $E_{1t_{i+1}}^{(i)} \rightarrow \mathcal{O}(\mu_{1,i+1}) \rightarrow 0$ be the quotient of minimal degree, then we define $E_1^{(i+1)}$ to be the elementary transform of $E_1^{(i)}$ along $\mathcal{O}(\mu_{1,i+1})$ at $X_{t_{i+1}}$, namely $E_1^{(i+1)}$ satisfies the exact sequence

$$0 \rightarrow E_1^{(i+1)} \rightarrow E_1^{(i)} \rightarrow_{X_{t_{i+1}}} \mathcal{O}(\mu_{1,i+1}) \rightarrow 0.$$

Let s_1 be the minimal integer such that $\Delta(E_1^{(s_1)}) = 0$. Then

$$\Delta(E_1) = 2s_1 - 4 \sum_{i=1}^{s_1} \mu_{1,i}, \quad (13)$$

where $\mu_{1,i} \leq 0$. Same as the proof of Proposition 4.3 in [6], we can show that

$$s_1 \geq \deg E_1 - 2 \deg f_* E_1 + 2 \dim H^0(\mathcal{O}(\mu_{1,s_1})). \quad (14)$$

Hence, by (12), (13) and (14), we have

$$\Delta(E) \geq -6 \deg f_* E_1 + 6 \dim H^0(\mathcal{O}(\mu_{1,s_1})) - 6\mu_{1,s_1} + 2. \quad (15)$$

On the other hand, we claim that $\deg f_* E_1 \leq -2$ when $\phi : B \rightarrow M$ passes through the generic points, which means that E_y is $(2, 0)$ -stable for generic $y \in B$. To see it, we consider

$$0 \rightarrow f^* f_* E_1 \rightarrow E_1 \rightarrow \mathcal{E}_1 \rightarrow 0$$

where \mathcal{E}_1 is locally free over $f^{-1}(C \setminus T)$ and $T \subset C$ is a finite set such that E_{1t} ($t \in T$) is not semi-stable. Thus, for any $y \in B$, the sequence

$$0 \rightarrow (f^* f_* E_1)_y \rightarrow E_{1y} \rightarrow \mathcal{E}_{1y} \rightarrow 0$$

is still exact, so we can consider $(f^* f_* E_1)_y$ as a sub line bundle of E_y . Then since E_y is $(2, 0)$ -stable of degree 1 for generic $y \in B$,

$$\deg f_* E_1 + 2 = \deg(f^* f_* E_1)_y + 2 < \frac{\deg E_y + 2}{3},$$

which implies $\deg f_* E_1 \leq -2$. Therefore, if $\mu_{1,s_1} < 0$, we have $\Delta(E) \geq 12 + 6 + 2 = 20$ by (15). If $\mu_{1,s_1} = 0$, by tensoring E with $\pi^* \mathcal{O}(-\mu_{1,s_1})$, we may assume $\dim H^0(\mathcal{O}(\mu_{1,s_1})) = 1$, then $\Delta(E) \geq 12 + 6 + 2 = 20$.

From now, we will assume that $\Delta(E_1) = 0$.

If $c_2(F_2) \neq 0$, then F_2 is not locally free, which implies that there is a $y_0 \in B$ such that $F_2|_{C \times \{y_0\}}$ has torsion $\tau(F_2|_{C \times \{y_0\}}) \neq 0$ since F_2 is B -flat (cf Lemma 1.27 of [10]). Let

$$0 \rightarrow \tau(F_2|_{C \times \{y_0\}}) \rightarrow F_2|_{C \times \{y_0\}} \rightarrow F_2^0 \rightarrow 0. \quad (16)$$

Then F_2^0 being a quotient line bundle of $E|_{C \times \{y_0\}}$ implies $\deg F_2^0 \geq 1$ since $E|_{C \times \{y_0\}}$ is stable. By sequences (11) and (16), we have

$$\deg E_1|_{C \times \{y_0\}} = 1 - \deg F_2^0 - \dim \tau(F_2|_{C \times \{y_0\}}) \leq -1$$

which, by the formula (12), implies that

$$\Delta(E) \geq 6c_2(F_2) + (4 - 6(-1))(\mu_1 - \mu_2) \geq 11.$$

When $\phi : B \rightarrow M$ passes through the generic points, means that $E_y = E|_{C \times \{y\}}$ is $(2, 0)$ -stable for generic $y \in B$. Which implies that $\deg E_1 \leq -1$.

If $\mu_1 = 0$. If $\deg E_1 \leq -2$, then $\Delta(E) \geq 6 + (4 - 6(-2)) = 22$. Now we assume that $\deg E_1 = -1$. Note that $c_2(F_2) \neq 0$ and F_2 being C -flat also imply that there exists a $t_0 \in C$ such that $F_2|_{X_{t_0}}$ has torsion $\tau(F_2|_{X_{t_0}}) \neq 0$. Let $0 \rightarrow \tau(F_2|_{X_{t_0}}) \rightarrow F_2|_{X_{t_0}} \rightarrow \mathcal{Q} \rightarrow 0$ and $E' = \ker(E \rightarrow_{X_{t_0}} \mathcal{Q})$, then

$$0 \rightarrow E' \rightarrow E \rightarrow_{X_{t_0}} \mathcal{Q} \rightarrow 0$$

which, for any $y \in B$, induces exact sequence

$$0 \rightarrow E'|_{C \times \{y\}} \rightarrow E|_{C \times \{y\}} \rightarrow_{(t_0, y)} \mathcal{Q} \rightarrow 0. \quad (17)$$

Thus all $E'|_{C \times \{y\}}$ are semi-stable of degree 0. If $\phi : B \rightarrow M$ passes through the generic points, means that $E_y = E|_{C \times \{y\}}$ is $(2, 0)$ -stable for generic $y \in B$, thus E'_y is stable by (17). This implies that $\Delta(E') > 0$. Otherwise $\{E'_y\}_{y \in B}$ are s-equivalent by applying Theorem 2.4 to $\pi : X \rightarrow B$, which implies $E' = f^*V \otimes \pi^*L$ for a stable bundle V on C and a line bundle L on B . Then $E_t = E'_t \cong L \oplus L \oplus L$ for any $t \neq t_0$, which is a contradiction since E is not semi-stable on the generic fiber of $f : X \rightarrow C$.

To compute $\Delta(E')$, consider the relative Harder-Narasimhan filtration

$$0 \rightarrow E'_1 \rightarrow E' \rightarrow F'_2 \rightarrow 0$$

over C , then $\mu(E'_1|_{X_t}) = \mu_1$, $\mu(F'_2|_{X_t}) = \mu_2$ for generic $t \in C$. Then

$$\Delta(E') = \frac{3}{2}\Delta(E'_1) + 6c_2(F'_2) - 6\deg E'_1(\mu_1 - \mu_2) \geq 12.$$

To see it, we can assume $\Delta(E'_1) = c_2(F'_2) = 0$ and $\deg E'_1 = -1$. Note that $E'_{1y} = E'_1|_{X_y}$ is stable of degree -1 for generic $y \in B$ since E'_y is stable. Then, by Theorem 2.4, there are vector bundles V'_1, V'_2 on C of rank 2, 1 respectively, and line bundles $\mathcal{O}(\mu_i)$ of degree μ_i ($i = 1, 2$) on B such that $E'_1 = f^*V'_1 \otimes \pi^*\mathcal{O}(\mu_1)$, $F'_2 = f^*V'_2 \otimes \pi^*\mathcal{O}(\mu_2)$. Then we have

$$0 \rightarrow f^*V'_1 \otimes \pi^*\mathcal{O}(\mu_1 - \mu_2) \rightarrow E' \otimes \pi^*\mathcal{O}(-\mu_2) \rightarrow f^*V'_2 \rightarrow 0$$

which defines a morphism $\psi : B \rightarrow \mathbb{P}$ to a projective space such that $\pi^*\mathcal{O}(\mu_1 - \mu_2) = \psi^*\mathcal{O}_{\mathbb{P}}(1)$. Thus $\mu_1 - \mu_2 \geq 2$ and $\Delta(E') \geq -6(-1)(\mu_1 - \mu_2) \geq 12$. Thus

$$\Delta(E) = \Delta(E') + 6(\mu(E_{t_0}) - \mu(\mathcal{Q})) \geq 12 + 6(\frac{2}{3} + 1) = 22.$$

If $\mu_1 = \frac{1}{2}$, i.e., E_1 is semi-stable of degree 1 on the generic fiber of $f : X \rightarrow C$. By Theorem 2.4, $\Delta(E_1) = 0$ implies that there exist a stable rank 2 vector bundle V of degree 1 on B and a line bundle L on C such that $E_1 = \pi^*V \otimes f^*L$. It is easy to see

$$\deg E_1 = 2\deg L \in 2\mathbb{Z}.$$

Moreover, we can show that $\deg E_1 \leq -4$. In fact, if $\deg E_1 = -2$, E_{1y} is stable of degree -2 since E_y is $(2, 0)$ -stable for generic $y \in B$. Applying Theorem 2.4, $\Delta(E_1) = 0$ implies that there exist a stable rank 2 vector bundle V_1 of degree -2 on C and a line bundle L_1 on B such that $E_1 = f^*V_1 \otimes \pi^*L_1$ and $\deg E_1|_{f^{-1}(t)} = 2\deg L_1$ for any $t \in C$, which imply a contradiction since $E_1|_{f^{-1}(t)}$ is semi-stable of degree 1 for generic $t \in C$. Thus $\deg E_1 \leq -4$ and $\Delta(E) \geq 6 + (4 - 6(-4)) \cdot \frac{1}{2} = 20$.

Now we assume that $c_2(F_2) = 0$ and $\Delta(E_1) = 0$. The assumption $c_2(F_2) = 0$ implies F_2 is locally free and there is a line bundle V_2 on C such that $F_2 = f^*V_2 \otimes \pi^*\mathcal{O}(\mu_2)$. The degree formula becomes

$$\Delta(E) = (4 - 6\deg E_1)(\mu_1 - \mu_2).$$

If $\mu_1 = 0$, then E_1 is semi-stable of degree 0 at every fiber of $f : X \rightarrow C$ by applying Theorem 2.4. If $\deg E_1 \leq -1$, $\Delta(E) \geq (4 - 6(-1)) = 10$. Now we consider the case $\deg E_1 = 0$. In this case E_1 is semi-stable of degree 0 at every fiber of $\pi : X \rightarrow B$ since E is stable of degree 1 at every fiber of $\pi : X \rightarrow B$. Apply Theorem 2.4 to $f : X \rightarrow C$ (resp. $\pi : X \rightarrow B$), then $\Delta(E_1) = 0$ implies that $\{E_{1y} := E_1|_{C \times \{y\}}\}_{y \in B}$ (resp. $\{E_{1t} := E_1|_{\{t\} \times B}\}_{t \in C}$) are semi-stable and isomorphic to each other. By tensoring E (thus E_1) with π^*L (where L is a line bundle of degree 0 on B), we can assume that $H^0(E_{1t}) \neq 0$ ($\forall t \in C$), which has dimension at most 2 since E_{1t} is semi-stable of degree 0. If $\dim H^0(E_{1t}) = 2$, then f_*E_1 is a degree 0 vector bundle of rank 2 on C , $E_1 = f^*f_*E_1$ and $E \otimes \pi^*\mathcal{O}(-\mu_2)$ fits an exact sequence

$$0 \rightarrow f^*f_*E_1 \otimes \pi^*\mathcal{O}(\mu_1 - \mu_2) \rightarrow E \otimes \pi^*\mathcal{O}(-\mu_2) \rightarrow f^*V_2 \rightarrow 0$$

which defines a morphism $\psi : B \rightarrow \mathbb{P}$ to a projective space \mathbb{P} such that $\psi^*\mathcal{O}_{\mathbb{P}}(1) = \mathcal{O}(\mu_1 - \mu_2)$, thus $\mu_1 - \mu_2 \geq 2$ and $\Delta(E) \geq 4 \cdot 2 = 8$. If $\dim H^0(E_{1t}) = 1$, f_*E_1 is a line bundle on C since $\{E_{1t}\}_{t \in C}$ are isomorphic each other. Then we have an exact sequence

$$0 \rightarrow f^*f_*E_1 \rightarrow E_1 \rightarrow f^*V_1 \otimes \pi^*L_1 \rightarrow 0$$

for a line bundle V_1 on C and a degree 0 line bundle L_1 on B . Consider $f^*f_*E_1$ as a sub line bundle of E and let $E' := E/(f^*f_*E_1)$, there is an induced morphism $\beta : f^*V_1 \otimes \pi^*L_1 \rightarrow E'$ satisfying the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & f^*f_*E_1 & \longrightarrow & E_1 & \longrightarrow & f^*V_1 \otimes \pi^*L_1 \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \beta \\ 0 & \longrightarrow & f^*f_*E_1 & \longrightarrow & E & \longrightarrow & E' \longrightarrow 0. \end{array}$$

By the snake lemma, β is injective and $\text{Coker}(\beta) = f^*V_2 \otimes \pi^*\mathcal{O}(\mu_2)$. Then $E' \otimes \pi^*\mathcal{O}(-\mu_2)$ fits an exact sequence

$$0 \rightarrow f^*V_1 \otimes \pi^*\mathcal{O}(-\mu_2) \rightarrow E' \otimes \pi^*\mathcal{O}(-\mu_2) \rightarrow f^*V_2 \rightarrow 0, \quad (18)$$

where $\mathcal{O}(-\mu_2)$ is a line bundle of degree $-\mu_2$ on B . (18) defines a morphism $\varphi' : B \rightarrow \mathbb{P}H^1(V_2^{-1} \otimes V_1) = \mathbb{P}^{g-1}$ such that $\varphi'^*\mathcal{O}_{\mathbb{P}^{g-1}}(1) = \mathcal{O}(-\mu_2)$, thus $-\mu_2 \geq 2$ and $\Delta(E) \geq 4 \cdot 2 = 8$.

If $\phi : B \rightarrow M$ passes through the generic points, then E_y is $(2, 0)$ -stable for generic $y \in B$. Then $\deg E_1 \leq -1$. If $\deg E_1 \leq -3$, it's easy to see that $\Delta(E) \geq 4 - 6(-3) = 22$. If $\deg E_1 = -1$ (or -2), then E_{1y} is stable of degree -1 (or -2) for generic $y \in B$ since E_y is $(2, 0)$ -stable, which implies that all the bundles $\{E_{1y} = E_1|_{C \times \{y\}}\}_{y \in B}$ are stable and isomorphic each other by applying Theorem 2.4 to $\pi : X \rightarrow B$. Then there is a stable vector bundle V_1 of rank 2 on C and a line bundle $\mathcal{O}(\mu_1)$ on B such that $E_1 = f^*V_1 \otimes \pi^*\mathcal{O}(\mu_1)$. Then $E \otimes \pi^*\mathcal{O}(-\mu_2)$ fits an exact sequence

$$0 \rightarrow f^*V_1 \otimes \pi^*\mathcal{O}(\mu_1 - \mu_2) \rightarrow E \otimes \pi^*\mathcal{O}(-\mu_2) \rightarrow f^*V_2 \rightarrow 0$$

which defines a morphism $\varphi : B \rightarrow \mathbb{P}'$ to a projective space \mathbb{P}' such that $\varphi^*\mathcal{O}_{\mathbb{P}'}(1) = \mathcal{O}(\mu_1 - \mu_2)$, thus $\mu_1 - \mu_2 \geq 2$ and $\Delta(E) \geq (4 - 6(-1)) \cdot 2 = 20$.

If $\mu_1 = \frac{1}{2}$, then E_1 is semi-stable of degree 1 at generic fiber of $f : X \rightarrow C$. By Applying Theorem 2.4, $\Delta(E_1) = 0$ implies $\{E_{1t} = E_1|_{\{t\} \times B}\}_{t \in C}$ are semi-stable of degree 1 and s -equivalent, thus they are stable and isomorphic. Then $E_1 = \pi^*V \otimes f^*L$ for a stable bundle V of degree 1 on B and a line bundle L on C , which implies that $\deg E_1 = 2\deg L \in 2\mathbb{Z}$ and $f_*E_1 = f_*(\pi^*V \otimes f^*L) \cong f_*\pi^*V \otimes L \cong L$. Then we have an exact

$$0 \rightarrow f^*L \rightarrow E_1 \rightarrow f^*L \otimes \pi^*\mathcal{O}(1) \rightarrow 0$$

for a degree 1 line bundle $\mathcal{O}(1)$ on B . Consider f^*L as a sub line bundle of E and let $E' := E/(f^*L)$, same as above, we have an exact sequence

$$0 \rightarrow f^*L \otimes \pi^*\mathcal{O}(1 - \mu_2) \rightarrow E' \otimes \pi^*\mathcal{O}(-\mu_2) \rightarrow f^*V_2 \rightarrow 0$$

and $1 - \mu_2 \geq 2$. Thus $\mu_2 \leq -1$ and $\Delta(E) \geq 4(\frac{1}{2} - (-1)) = 6$.

If $\phi : B \rightarrow M$ passes through the generic points, $\deg E_1 \leq -2$ since $\deg E_1 \in 2\mathbb{Z}$ and E_y is $(2, 0)$ -stable for generic $y \in B$. Thus $\Delta(E) \geq (4 - 6(-2))(\frac{1}{2} - (-1)) = 24$. \square

Lemma 4.4. *If $g \geq 4$, M contains $(1, 2)$ -stable points. If $g > 4$, M contains $(3, 1)$ -stable points.*

Proposition 4.5. *When $n = 2$ and $\text{rk} E_1 = 1$, $\Delta(E) \geq 6$. If $g > 4$ and $\phi : B \rightarrow M$ passes through the generic points, then $\Delta(E) \geq 18$.*

Proof. In this case, $E_1 = f^*V_1 \otimes \pi^*\mathcal{O}(\mu_1)$ for a line bundle V_1 on C and a degree μ_1 line bundle $\mathcal{O}(\mu_1)$ on B . The degree formula becomes

$$\Delta(E) = \frac{3}{2}\Delta(F_2) + (2 - 6\deg E_1)(\mu_1 - \mu_2). \quad (19)$$

Tensoring E with π^*L for some suitable line bundle L on B , we can assume that $\mu_2 = 0$ or $\frac{1}{2}$.

We consider the case $\Delta(F_2) = 0$ at first, which implies that F_2 is locally free and F_2 is semi-stable of slope μ_2 at every fiber of $f : X \rightarrow C$.

If $\mu_2 = 0$, then F_2 is semi-stable of degree 0 at generic fiber of $f : X \rightarrow C$. By applying Theorem 2.4, $\Delta(F_2) = 0$ implies all the bundles $\{F_{2y} := F_2|_{C \times \{y\}}\}_{y \in B}$ are isomorphic to each other. If $\deg E_1 \leq -1$, it's easy to see $\Delta(E) \geq (2 - 6(-1))(\mu_1 - \mu_2) \geq 8$. If $\deg E_1 = 0$, F_{2y} is

stable of degree 1 for any $y \in B$, then there is a stable vector bundle V_2 of degree 1 on C and a degree μ_2 line bundle $\mathcal{O}(\mu_2)$ on B such that $F_2 = f^*V_2 \otimes \pi^*\mathcal{O}(\mu_2)$, and $E \otimes \mathcal{O}(-\mu_2)$ satisfies an exact sequence

$$0 \rightarrow f^*V_1 \otimes \pi^*\mathcal{O}(\mu_1 - \mu_2) \rightarrow E \otimes \mathcal{O}(-\mu_2) \rightarrow f^*V_2 \rightarrow 0$$

which defines a morphism

$$\varphi : B \rightarrow \mathbb{P}H^1(V_2^{-1} \otimes V_1) \cong \mathbb{P}^{2g-2}$$

such that $\varphi^*\mathcal{O}_{\mathbb{P}^{2g-2}}(1) = \mathcal{O}(\mu_1 - \mu_2)$ and $\phi : B \rightarrow M$ factors through $\varphi : B \rightarrow \mathbb{P}H^1(V_2^{-1} \otimes V_1) \cong \mathbb{P}^{2g-2}$, which implies that the normalization of $\varphi(B)$ is an elliptic curve. Hence $\mu_1 - \mu_2 \geq 3$ and $\Delta(E) \geq 2 \cdot 3 = 6$.

When $\phi : B \rightarrow M$ passes through a generic point, i.e., there is a $y_0 \in B$ such that E_{y_0} is (1,1)-stable, which implies that $\deg E_1 \leq -1$. If $\deg E_1 \leq -3$, we have $\Delta(E) \geq (2 - 6(-3))(\mu_1 - \mu_2) \geq 20$. When $\deg E_1 = -1$ or -2 , F_{2y_0} is stable of degree 2(or 3) since E_{y_0} is (1,1)-stable. And since all the bundles $\{F_{2y} := F_2|_{C \times \{y\}}\}_{y \in B}$ are isomorphic to each other, then there is a stable vector bundle V_2 of degree 2(or 3) on C and a degree μ_2 line bundle $\mathcal{O}(\mu_2)$ on B such that $F_2 = f^*V_2 \otimes \pi^*\mathcal{O}(\mu_2)$. Same as above, we can see that $\mu_1 - \mu_2 \geq 3$ and $\Delta(E) \geq (2 - 6(-1)) \cdot 3 = 24$.

If $\mu_2 = \frac{1}{2}$. By applying Theorem 2.4, $\Delta(F_2) = 0$ implies all the bundles $\{F_{2t} = F_2|_{\{t\} \times B}\}_{t \in C}$ are semi-stable of degree 1 and s -equivalent, thus they are stable and isomorphic to each other. Then $F_2 = \pi^*V \otimes f^*L_2$ for a stable bundle V of degree 1 on B and a line bundle L_2 on C , which implies that $\deg F_2 = 2\deg L_2 \in 2\mathbb{Z}$ and

$$f_*f^*F_2 = f_*f^*(\pi^*V \otimes f^*L_2) \cong f^*(f_*\pi^*V \otimes L_2) \cong f^*L_2.$$

Then we have exact sequence

$$0 \rightarrow f_*f^*F_2 = f^*L_2 \rightarrow F_2 \rightarrow f^*L_2 \otimes \pi^*\mathcal{O}(1) \rightarrow 0$$

for a degree 1 line bundle $\mathcal{O}(1)$ on B . Consider $f^*L_2 \otimes \pi^*\mathcal{O}(1)$ as a quotient line bundle of E and let $E' := \ker(E \rightarrow f^*L_2 \otimes \pi^*\mathcal{O}(1))$, there is a induced morphism $\gamma : E' \rightarrow f_*f^*F_2 = f^*L_2$ satisfying the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & f^*L_2 \otimes \pi^*\mathcal{O}(1) \longrightarrow 0 \\ & & \gamma \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & f^*L_2 & \longrightarrow & F_2 & \longrightarrow & f^*L_2 \otimes \pi^*\mathcal{O}(1) \longrightarrow 0. \end{array}$$

By the snake lemma, γ is surjective and $\ker \gamma = E_1 = f^*V_1 \otimes \pi^*\mathcal{O}(\mu_1)$. Then E' fits an exact sequence

$$0 \rightarrow f^*V_1 \otimes \pi^*\mathcal{O}(\mu_1) \rightarrow E' \rightarrow f^*L_2 \rightarrow 0$$

which defines a morphism

$$\varphi_{E'} : B \rightarrow \mathbb{P}H^1(L_2^{-1} \otimes V_1) \cong \mathbb{P}^{\frac{1-3d_1}{2}+g-2}$$

such that $\varphi_{E'}^*\mathcal{O}_{\mathbb{P}H^1(L_2^{-1} \otimes V_1)}(1) = \mathcal{O}(\mu_1)$. Thus $\mu_1 \geq 2$. On the other hand, since E is stable of degree 1 at every fiber of $\pi : X \rightarrow B$ and $\deg F_2 = 2\deg L_2 \geq 1$, we have $\deg F_2 \geq 2$. Hence, $\deg E_1 \leq -1$ and

$$\Delta(E) \geq (2 - 6(-1))(2 - \frac{1}{2}) = 12.$$

If $\phi : B \rightarrow M$ passes a generic point, there is a $y_0 \in B$ such that E_{y_0} is (1,1)-stable, which implies

$$\deg L_2 - 1 > \frac{\deg E + 1 - 1}{3} = \frac{1}{3}.$$

So $\deg F_2 = 2\deg L_2 \geq 4$ and $\deg E_1 \leq -3$. Thus

$$\Delta(E) \geq (2 - 6(-3))(2 - (\frac{1}{2})) = 30.$$

From now we will consider the case that $\Delta(F_2) = 4c_2(F_2) - c_1(F_2)^2 \neq 0$.

We consider the case that F_2 is locally free at first.

If $\mu_2 = 0$, then F_2 is semi-stable of degree 0 at the generic fiber of $f : X \rightarrow C$. By Theorem 2.4, $\Delta(F_2) \neq 0$ implies $\Delta(F_2) > 0$. On the other hand, $c_1(F_2)^2 = 0$ since F_2 is semi-stable of degree 0 at the generic fiber of $f : X \rightarrow C$ and $\text{Pic}(C \times B) = \text{Pic}(C) \times \text{Pic}B$. Thus $\Delta(F_2) = 4c_2(F_2) \geq 4$ and

$$\Delta(E) \geq \frac{3}{2} \cdot 4 + 2 \cdot (\mu_1 - \mu_2) \geq 8.$$

When $\phi : B \rightarrow M$ passes through a generic point, i.e., a $(1, 1)$ -stable point, we have $\deg E_1 \leq -1$. If $\deg E_1 \leq -2$, it's easy to see $\Delta(E) \geq \frac{3}{2} \cdot 4 + (2 - 6(-2))(\mu_1 - \mu_2) \geq 20$. Now we assume that $\deg E_1 = -1$, then F_{2y} is stable of degree 2 for generic $y \in B$ since E_y is $(1, 1)$ -stable. We claim that $\Delta(F_2) \geq 8$, which implies $\Delta(E) \geq \frac{3}{2} \cdot 8 + (2 - 6(-1))(\mu_1 - \mu_2) \geq 20$ by (19).

We prove the above claim as following: If F_2 is semi-stable of degree 0 at every fiber $f : X \rightarrow C$, then F_2 induces a non-trivial morphism $\varphi_{F_2} : C \rightarrow \mathbb{P}^1$ (cf. [12]) such that $\varphi_{F_2}^* \mathcal{O}_{\mathbb{P}^1}(1) = (\det f_! F_2)^{-1}$ which has degree $c_2(F_2)$ by Grothendieck-Riemann-Roch theorem. Then

$$\Delta(F_2) = 4c_2(F_2) = 4\deg \varphi_{F_2} \geq 8.$$

If there exists a $t_0 \in C$ such that $F_{2t_0} = F_2|_{f^{-1}(t_0)}$ is not semi-stable, let $F_{2t_0} \rightarrow \mathcal{O}(\mu) \rightarrow 0$ be the quotient line bundle of minimal degree $\mu < 0$ and $F'_2 = \ker(F_2 \rightarrow_{X_{t_0}} \mathcal{O}(\mu) \rightarrow 0)$. If $\Delta(F'_2) \neq 0$, then $\Delta(F'_2) = 4c_2(F'_2) \geq 4$ and $\Delta(F_2) = \Delta(F'_2) - 4\mu \geq 8$ by Lemma 2.5. If $\Delta(F'_2) = 0$, all the bundles $\{F'_{2y} = F_2|_{C \times \{y\}}\}_{y \in B}$ are isomorphic to each other by applying Theorem 2.4 to $f : X \rightarrow C$. On the other hand, by the definition of F'_2 , we have exact sequences

$$0 \rightarrow F'_2 \rightarrow F_2 \rightarrow_{X_{t_0}} \mathcal{O}(\mu) \rightarrow 0 \quad (20)$$

and

$$0 \rightarrow F'_{2y} \rightarrow F_{2y} \rightarrow_{(t_0, y)} \mathbb{C} \rightarrow 0 \text{ for any } y \in B.$$

Then F'_{2y} is stable of degree 1 for generic $y \in B$ since F_{2y} is stable of degree 2. Thus all the bundles $\{F'_{2y} = F_2|_{C \times \{y\}}\}_{y \in B}$ are stable of degree 1 and isomorphic to each other, then $F'_2 = f^* V'_2 \otimes \pi^* L'$ for a degree 1 stable vector bundle V'_2 on C and a degree 0 line bundle L' on B . Then (20) induces a non-trivial morphism $\psi : B \rightarrow \mathbb{P}(V_{2t_0}^{'*})$ such that $\mathcal{O}(-\mu) = \psi^* \mathcal{O}_{\mathbb{P}(V_{2t_0}^{'*})}(1)$. Thus $-\mu \geq 2$ and $\Delta(F_2) \geq 8$.

If $\mu_2 = \frac{1}{2}$, then F_2 is semi-stable of degree 1 at the generic fiber of $f : X \rightarrow C$. It's known that there is a unique stable rank 2 vector bundle with fixed determinant of degree 1 on an elliptic curve, and note that F_2 is locally free. Thus $\Delta(F_2) > 0$ if and only if there exists $t_1 \in C$ such that $F_{2t_1} := F_2|_{\{t_1\} \times B}$ is not semi-stable.

Let $F_{2t_1} \rightarrow \mathcal{O}(\mu_{2,1}) \rightarrow 0$ be the quotient of minimal degree and

$$0 \rightarrow F_2^{(1)} \rightarrow F_2 \rightarrow_{X_{t_1}} \mathcal{O}(\mu_{2,1}) \rightarrow 0$$

be the elementary transformation of F_2 along $\mathcal{O}(\mu_{2,1})$ at X_{t_1} . If $F_2^{(i)}$ is defined and $\Delta(F_2^{(i)}) > 0$, let $t_{i+1} \in C$ such that $F_{2t_{i+1}}^{(i)} := F_2^{(i)}|_{X_{t_{i+1}}}$ is not semi-stable and $F_{2t_{i+1}}^{(i)} \rightarrow \mathcal{O}(\mu_{2,i+1}) \rightarrow 0$ be the quotient of minimal degree, then we define $F_2^{(i+1)}$ to be the elementary transform of $F_2^{(i)}$ along $\mathcal{O}(\mu_{2,i+1})$ at $X_{t_{i+1}}$, namely $F_2^{(i+1)}$ satisfies the exact sequence

$$0 \rightarrow F_2^{(i+1)} \rightarrow F_2^{(i)} \rightarrow_{X_{t_{i+1}}} \mathcal{O}(\mu_{2,i+1}) \rightarrow 0. \quad (21)$$

Let s_2 be the minimal integer such that $\Delta(F_2^{(s_2)}) = 0$. Then

$$\Delta(F_2) = \Delta(F_2^{(s_2)}) + \sum_{i=1}^{s_2} (2 - 4\mu_{2,i}) = 2s_2 - 4 \sum_{i=1}^{s_2} \mu_{2,i},$$

where $\mu_{2,i} \leq 0 (i = 1, 2, \dots, s_2)$. Take direct image of (21), we have

$$0 \rightarrow f_* F_2^{(s_2)} \rightarrow f_* F_2^{(s_2-1)} \rightarrow_{t_{s_2}} H^0(\mathcal{O}(\mu_{2,s_2})) \rightarrow R^1 f_* F_2^{(s_2)} = 0$$

and $\deg f_* F_2^{(i+1)} \leq \deg f_* F_2^{(i)}$, which implies

$$\deg f_* F_2^{(s_2)} \leq \deg f_* F_2 - \dim H^0(\mathcal{O}(\mu_{2,s_2})). \quad (22)$$

Restrict (21) to a fiber $X_y = \pi^{-1}(y)$, we have exact sequence

$$0 \rightarrow F_{2y}^{(i+1)} \rightarrow F_{2y}^{(i)} \rightarrow_{(t_{i+1}, y)} \mathbb{C} \rightarrow 0$$

which implies that

$$\deg F_{2y}^{(s_2)} = \deg F_{2y}^{(s_2-1)} - 1 = \dots = \deg F_{2y} - s_2 = \deg F_2 - s_2. \quad (23)$$

On the other hand, by Theorem 2.4, $\Delta(F_2^{(s_2)}) = 0$ implies that there exists a stable rank 2 vector bundle V of degree 1 on B and a line bundle L on C such that $F_2^{(s_2)} = \pi^* V \otimes f^* L$. It's easy to see

$$\deg F_{2y}^{(s_2)} = 2 \deg L = 2 \deg f_* F_2^{(s_2)}.$$

Thus combine (22) and (23), we have the inequality

$$s_2 \geq 1 - \deg E_1 - 2 \deg f_* F_2 + 2 \dim H^0(\mathcal{O}(\mu_{2,s_2})). \quad (24)$$

We claim that $\deg f_* F_2 \leq -\deg E_1$. To see it, consider

$$0 \rightarrow \mathcal{F}'_2 := f^* f_* F_2 \rightarrow F_2 \rightarrow \mathcal{F}_2 \rightarrow 0 \quad (25)$$

where \mathcal{F}_2 is locally free on $f^{-1}(C \setminus T)$ and $T \subset C$ is a finite set such that $F_{2t} (t \in T)$ is not semi-stable. Thus, for any $y \in B$, the sequence

$$0 \rightarrow \mathcal{F}'_{2y} \rightarrow F_{2y} \rightarrow \mathcal{F}_{2y} \rightarrow 0 \quad (26)$$

is still exact, which implies that \mathcal{F}_2 is B -flat (cf Lemma 2.1.4 of [7]). The sequence (26) and (11) already imply $\deg f_* F_2 = \deg \mathcal{F}'_{2y} \leq -\deg E_1$ since E_y is stable of degree 1 (Choose $y \in B$ such that \mathcal{F}_{2y} is free, then $\deg \mathcal{F}_{2y} = \mu(\mathcal{F}_{2y}) > \mu(E_y) = \frac{1}{3}$ and $\deg f_* F_2 = \deg \mathcal{F}'_{2y} = \deg F_{2y} - \deg \mathcal{F}_{2y} \leq 1 - \deg E_1 - 1 = -\deg E_1$). Thus

$$\begin{aligned} \Delta(E) &= \frac{3}{2} \Delta(F_2) + (2 - 6 \deg E_1)(\mu_1 - \mu_2) \\ &= \frac{3}{2} (2s_2 - 4 \sum_{i=1}^{s_2} \mu_{2,i}) + (2 - 6 \deg E_1)(\mu_1 - \mu_2) \\ &\geq 3(1 - \deg E_1 - 2 \deg f_* F_2 + 2 \dim H^0(\mathcal{O}(\mu_{2,s_2}))) - 2 \sum_{i=1}^{s_2} \mu_{2,i} + (2 - 6 \deg E_1)(\mu_1 - \mu_2) \\ &\geq 3(1 - \deg E_1 + 2 \deg E_1 + 2 \dim H^0(\mathcal{O}(\mu_{2,s_2}))) - 2 \sum_{i=1}^{s_2} \mu_{2,i} + (2 - 6 \deg E_1) \frac{1}{2} \\ &\geq 4 + 6 \dim H^0(\mathcal{O}(\mu_{2,s_2})) - 6 \mu_{2,s_2}. \end{aligned}$$

If $\mu_{2,s_2} < 0$, then $\Delta(E) \geq 4 - 6(-1) = 10$. If $\mu_{2,s_2} = 0$, tensoring E with $\pi^* \mathcal{O}(-\mu_{2,s_2})$ we can assume that $\dim H^0(\mathcal{O}(\mu_{2,s_2})) \neq 0$ and $\Delta(E) \geq 4 + 6 = 10$.

When $\phi : B \rightarrow M$ passes the generic points, i.e., the (1,2)-stable points, the sequences (26) and (11) also imply that $\deg f_* F_2 \leq -2 - \deg E_1$ since E_y is (1,2)-stable of degree 1 for generic $y \in B$. Thus $\Delta(E) \geq 22$.

Now we consider the case that F_2 is not locally free, which implies there exists a $y_0 \in B$ such that $F_2|_{X_{y_0}}$ has torsion $\tau(F_2|_{X_{y_0}}) \neq 0$ since F_2 is B -flat (cf Lemma 1.27 of [10]). Let

$$0 \rightarrow \tau(F_2|_{X_{y_0}}) \rightarrow F_2|_{X_{y_0}} \rightarrow F_2^0 \rightarrow 0. \quad (27)$$

Then F_2^0 being quotient bundle of $E|_{X_{y_0}}$ implies that

$$\frac{\deg F_2^0}{2} = \mu(F_2^0) > \mu(E|_{X_{y_0}}) = \frac{1}{3} \implies \deg F_2^0 \geq 1$$

since $E|_{X_{y_0}}$ is stable of degree 1. By sequences (11) and (27), we have

$$\mu(E_1) = \deg E|_{X_{y_0}} = 1 - \deg F_2^0 - \dim \tau(F_2|_{X_{y_0}}) \leq -1,$$

which, by the formula (19), we have

$$\Delta(E) = \frac{3}{2}\Delta(F_2) + (2 - 6\deg E_1)(\mu_1 - \mu_2) \geq \frac{3}{2} \cdot 2 + (2 - 6(-1))\frac{1}{2} = 7.$$

When $\phi : B \rightarrow M$ passes through a generic point, in order to show $\Delta(E) \geq 18$, we note that F_2 being not locally free and C -flat also imply that there is a $t_0 \in C$ such that $F_2|_{X_{t_0}}$ has non-trivial torsion $\tau(F_2|_{X_{t_0}}) \neq 0$. Let $0 \rightarrow \tau(F_2|_{X_{t_0}}) \rightarrow F_2|_{X_{t_0}} \rightarrow \mathcal{Q} \rightarrow 0$ and $E' = \ker(E \rightarrow_{X_{t_0}} \mathcal{Q} \rightarrow 0)$, then

$$0 \rightarrow E' \rightarrow E \rightarrow_{X_{t_0}} \mathcal{Q} \rightarrow 0$$

which, for any $y \in B$, induces exact sequence

$$0 \rightarrow E'_y \rightarrow E_y \rightarrow_{(t_0, y)} \mathbb{C}^2 \rightarrow 0. \quad (28)$$

Thus all $E'_y := E'|_{C \times \{y\}}$ is of degree -1. If $\phi : B \rightarrow M$ passes through generic points, ie., E_y is (1,2)-stable for generic points $y \in B$, then it's easy to see that E'_y is stable for generic $y \in B$ by (28). This implies that $\Delta(E') > 0$. Otherwise $\{E'_y\}_{y \in B}$ are s-equivalent by applying Theorem 2.4 to $\pi : X \rightarrow B$, which implies $E' = f^*V' \otimes \pi^*L$ for a stable bundle V' on C and a line bundle L on B . Then $E_t = E'_t = L \oplus L \oplus L$ for any $t \neq t_0$, which is a contradiction since E is not semi-stable on the generic fiber of $f : X \rightarrow C$.

To compute $\Delta(E')$, we consider the relative Harder-Narasimhan filtration

$$0 \rightarrow E'_1 \rightarrow E' \rightarrow F'_2 \rightarrow 0$$

over C , then $\mu(E'_1|_{X_t}) = \mu_1$ and $\mu(F'_2|_{X_t}) = \mu_2$ for generic $t \in C$. Then

$$\Delta(E') = \frac{3}{2}\Delta(F'_2) + (-2 - 6\deg E'_1)(\mu_1 - \mu_2). \quad (29)$$

If $\mu_2 = 0$, then F'_2 is semi-stable of degree 0 at generic fiber of $f : X \rightarrow C$. We can prove that $\Delta(E') \geq 8$. It's easy to see that $\Delta(E') \geq (-2 - 6(-2))(\mu_1 - \mu_2) = 10$ when $\deg E'_1 \leq -2$, now we assume that $\deg E'_1 = -1$ and then F'_{2y} is semi-stable of degree 0 for generic $y \in B$ since E'_y is stable of degree -1. If $\Delta(F'_2) \neq 0$, then $\Delta(F'_2) = 4c_2(F'_2) \geq 4$ by Theorem 2.4 and the fact that $c_1(F'_2)^2 = 0$ since $\text{Pic}(C \times B) = \text{Pic}(C) \times \text{Pic}(B)$. Thus $\Delta(E') = \frac{3}{2}\Delta(F'_2) + (-2 - 6\deg E'_1)(\mu_1 - \mu_2) \geq \frac{3}{2} \cdot 4 + (-2 - 6(-1)) = 10$. If $\Delta(F'_2) = 0$, by applying Theorem 2.4 to $f : X \rightarrow C$ and $\pi : X \rightarrow B$, all the bundles $\{F'_{2t} := F'_2|_{\{t\} \times B}\}_{t \in C}$ are semi-stable and isomorphic to each other. Tensoring E (thus E') with π^*L (for a degree 0 line bundle L on B), we can assume that $H^0(F'_{2t}) \neq 0$ (for any $t \in C$), which has dimension at most 2 since F'_{2t} is semi-stable of degree 0. If $\dim H^0(F'_{2t}) = 2$, then $f_*F'_2 = V'_2$ is a vector bundle of rank 2 and $F'_2 = f^*V'_2$. Thus E' satisfies an exact sequence

$$0 \rightarrow f^*V'_1 \otimes \pi^*\mathcal{O}(\mu_1) \rightarrow E' \rightarrow f^*V'_2 \rightarrow 0$$

for a line bundle V'_1 on C and a degree μ_1 line bundle $\mathcal{O}(\mu_1)$ on B . Which induces a non-trivial morphism $\varphi_{E'} : B \rightarrow \mathbb{P}$ to a projective space \mathbb{P} such that $\mathcal{O}(\mu_1) = \varphi_{E'}^*\mathcal{O}_{\mathbb{P}}(1)$, thus $\mu_1 \geq 2$ and

$\Delta(E') \geq (-2 - 6(-1)) \cdot 2 = 8$. If $\dim H^0(F'_{2t}) = 1$, then $V'_2 := f_* F'_2$ is a line bundle and we have an exact sequence

$$0 \rightarrow f^* V'_2 \rightarrow F'_2 \rightarrow f^* V'_3 \otimes \pi^* \mathcal{O}(\mu_2) \rightarrow 0$$

for a line bundle V'_3 on C and a degree $\mu_2 = 0$ line bundle $\mathcal{O}(\mu_2)$ on B . Consider $f^* V'_3 \otimes \pi^* \mathcal{O}(\mu_2)$ as a quotient line bundle of E' and let $E'' := \ker(E' \rightarrow f^* V'_3 \otimes \pi^* \mathcal{O}(\mu_2) \rightarrow 0)$, then there is an induced morphism $\alpha'' : E'' \rightarrow f^* V'_2$ satisfies the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E'' & \longrightarrow & E' & \longrightarrow & f^* V'_3 \otimes \pi^* \mathcal{O}(\mu_2) \longrightarrow 0 \\ & & \alpha'' \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & f^* V'_2 & \longrightarrow & F'_2 & \longrightarrow & f^* V'_3 \otimes \pi^* \mathcal{O}(\mu_2) \longrightarrow 0. \end{array}$$

By the snake lemma, α'' is surjective and $\ker(\alpha'') = E'_1 = f^* V'_1 \otimes \pi^* \mathcal{O}(\mu_1)$. Thus E'' satisfies an exact sequence

$$0 \rightarrow f^* V'_1 \otimes \pi^* \mathcal{O}(\mu_1) \rightarrow E'' \rightarrow f^* V'_2 \rightarrow 0,$$

which induces a morphism $\varphi_{E''} : B \rightarrow \mathbb{P}^n$ to a projective space \mathbb{P}^n such that $\mathcal{O}(\mu_1) = \varphi_{E''}^* \mathcal{O}_{\mathbb{P}^n}(1)$. Thus $\mu_1 \geq 2$ and $\Delta(E') \geq (-2 - 6(-1)) \cdot 2 = 8$. Hence $\Delta(E') \geq 8$ when $\mu_2 = 0$. Then

$$\Delta(E) = \Delta(E') + 6(\mu(E_{t_0}) - \mu(\mathcal{Q})) \operatorname{rk} \mathcal{Q} \geq 8 + 12\left(\frac{1}{3} + \frac{1}{2}\right) = 18.$$

If $\mu_2 = \frac{1}{2}$, then F'_2 is semi-stable of degree 1 at generic fiber of $f : X \rightarrow C$. If $\Delta(F'_2) = 0$, then $F'_2 = \pi^* V' \otimes f^* L'_2$ for a rank 2 stable bundle V' on B and a line L'_2 on C . So $f_* F'_2 = f_*(\pi^* V' \otimes f^* L'_2) \cong f_* \pi^* V' \otimes L'_2 \cong L'_2$ is a line bundle and we have an exact sequence

$$0 \rightarrow f^* L'_2 \rightarrow F'_2 \rightarrow f^* L'_2 \otimes \pi^* \mathcal{O}(1) \rightarrow 0$$

for a degree 1 line bundle $\mathcal{O}(1)$ on B . Same as above, we can show that $\mu_1 \geq 2$ and $\Delta(E') \geq (-2 - 6(-1))(2 - \frac{1}{2}) = 6$. Then $\Delta(E) = \Delta(E') + 6(\mu(E_{t_0}) - \mu(\mathcal{Q})) \operatorname{rk} \mathcal{Q} \geq 6 + 12(\frac{1}{2} + \frac{1}{2}) = 18$. If $\Delta(F'_2) = 4c_2(F'_2) - c_1(F'_2)^2 \neq 0$, then $\Delta(F'_2) \geq 2$ by Theorem 2.4 and $c_1(F'_2)^2 = 2 \deg F'_2 (2\mu_2)$ since $\operatorname{Pic}(C \times B) = \operatorname{Pic}(C) \times \operatorname{Pic}(B)$. If $\phi : B \rightarrow M$ passes through the generic points when $g > 4$, then E_y is (3,1)-stable for generic $y \in B$, which implies that $\deg E'_1 \leq -3$. Thus $\Delta(E') \geq \frac{3}{2} \cdot 2 + (-2 - 6(-3))(\mu_1 - \mu_2) \geq 11$ and

$$\Delta(E) = \Delta(E') + 6(\mu(E_{t_0}) - \mu(\mathcal{Q})) \operatorname{rk} \mathcal{Q} \geq 11 + 12\left(\frac{1}{6} + \frac{1}{2}\right) = 19.$$

□

Now we consider the case that $n=1$, i.e., E is semi-stable on the generic fiber of $f : X \rightarrow C$. Tensoring E with $\pi^* L$ for a suitable line bundle L on B , we can assume that $0 \leq \deg(E|_{X_t}) \leq 2$ on $X_t = f^{-1}(t)$.

Proposition 4.6. *When E is semi-stable of degree 1 on the generic fiber of $f : X \rightarrow C$, we have $\Delta(E) \geq 14$. If $g \geq 4$ and when $\phi : B \rightarrow M$ passes through a generic point, then $\Delta(E) \geq 20$.*

Proof. It's known that there is a unique stable rank 3 vector bundle with a fixed determinant of degree 1 on an elliptic curve. Thus $\Delta(E) > 0$ if and only if there exists $t_1 \in C$ such that $E_{t_1} = E|_{X_{t_1}}$ is not semistable.

Let $E_{t_1} \rightarrow G_1 \rightarrow 0$ be an indecomposable quotient of minimal slope and

$$0 \rightarrow E^{(1)} \rightarrow E \rightarrow_{X_{t_1}} G_1 \rightarrow 0$$

be the elementary transformation of E along G_1 at X_{t_1} . If $E^{(i)}$ is defined and $\Delta(E^{(i)}) > 0$, let $t_{i+1} \in C$ such that $E_{t_{i+1}}^{(i)} = E^{(i)}|_{X_{t_{i+1}}}$ is not semi-stable and $E_{t_{i+1}}^{(i)} \rightarrow G_{i+1} \rightarrow 0$ be an indecomposable

quotient of minimal slop, then we define $E^{(i+1)}$ to be the elementary transformation of $E^{(i)}$ along G_{i+1} at $X_{t_{i+1}}$, namely $E^{(i+1)}$ satisfies the exact sequence

$$0 \rightarrow E^{(i+1)} \rightarrow E^{(i)} \rightarrow_{X_{t_{i+1}}} G_{i+1} \rightarrow 0. \quad (30)$$

Let s be the minimal integer such that $\Delta(E^{(s)}) = 0$, and let

$$s_1 = \#\{i : \text{rk} G_i = 1 \text{ but } i \neq s\} \quad \text{and} \quad s_2 = \#\{i : \text{rk} G_i = 2 \text{ but } i \neq s\}.$$

Then

$$s_1 + s_2 + 1 = s \quad \text{and} \quad s_1 + 2s_2 + \text{rk} G_s = \sum_{i=1}^s \text{rk} G_i,$$

and

$$\Delta(E) = \sum_{i=1}^s 6\left(\frac{1}{3} - \mu(G_i)\right) \text{rk} G_i \geq 2s_1 + 4s_2 + 6\left(\frac{1}{3} - \mu(G_s)\right) \text{rk} G_s, \quad (31)$$

where $\mu(G_i) \leq 0$ ($i = 1, 2, \dots, s$). Take direct image of (30), we have

$$0 \rightarrow f_* E^{(s)} \rightarrow f_* E^{(s-1)} \rightarrow_{t_s} H^0(G_s) \rightarrow 0 \quad (32)$$

(since $R^1 f_* E^{(s)} = 0$) and $\deg f_* E^{(i+1)} \leq \deg f_* E^{(i)}$, which imply

$$\deg f_* E^{(s)} \leq \deg f_* E - \dim H^0(G_s). \quad (33)$$

Restrict (30) to a fiber $X_y = \pi^{-1}(y)$, we have exact sequence

$$0 \rightarrow E_y^{(i+1)} \rightarrow E_y^{(i)} \rightarrow_{(t_{i+1}, y)} \mathbb{C}^{\text{rk} G_{i+1}} \rightarrow 0,$$

which implies that

$$\deg E_y^{(s)} = \deg E_y^{(s-1)} - \text{rk} G_s = \dots = \deg E_y - \sum_{i=1}^s \text{rk} G_i. \quad (34)$$

On the other hand, by Theorem 2.4, $\Delta(E^{(s)}) = 0$ implies that there exists a stable vector bundle V of rank 3 and degree 1 on B and a line bundle L on C such that $E^{(s)} = \pi^* V \otimes f^* L$. It's easy to see

$$\deg E_y^{(s)} = 3 \deg L = 3 \deg f_* E^{(s)}.$$

Thus combine (33) and (34), we have the inequality

$$\sum_{i=1}^s \text{rk} G_i \geq 1 - 3 \deg f_* E + 3 \dim H^0(G_s). \quad (35)$$

To see $\Delta(E) \geq 14$, consider the exact sequence

$$0 \rightarrow \mathcal{F}' = f^* f_* E \rightarrow E \rightarrow \mathcal{F} \rightarrow 0 \quad (36)$$

where \mathcal{F} is locally free on $f^{-1}(C \setminus T)$ and $T \subset C$ is a finite set such that E_t ($t \in T$) is not semi-stable. Thus, $\forall y \in B$, the sequence

$$0 \rightarrow \mathcal{F}'_y \rightarrow E_y \rightarrow \mathcal{F}_y \rightarrow 0 \quad (37)$$

is still exact, which implies \mathcal{F} is B -flat (cf Lemma 2.1.4 of [7]). The sequence (37) already implies $\deg f_* E = \deg \mathcal{F}'_y \leq 0$ since E_y is stable of degree 1.

If $\deg f_* E = \deg \mathcal{F}'_y = 0$, then \mathcal{F}_y is stable of degree 1 and \mathcal{F} is locally free (Otherwise, there is at least a $y_0 \in B$ such that \mathcal{F}_{y_0} has torsion (cf Lemma 1.27 of [10]). The stability of E_{y_0} implies that $\mathcal{F}_{y_0}/\text{torsion}$ has degree at least 1. Thus $\deg \mathcal{F}_{y_0} \geq 2$ and $\deg f_* E = \deg \mathcal{F}'_{y_0} \leq -1$, which contradicts to the assumption that $\deg f_* E = 0$). On the other hand, by the definition of \mathcal{F} , we

know that \mathcal{F} is semi-stable of degree 1 on the generic fiber of $f : X \rightarrow C$. This implies $\Delta(\mathcal{F}) > 0$. (Otherwise, $\{\mathcal{F}_t\}_{t \in C}$ are semi-stable of degree 1 and s-equivalent by applying Theorem 2.4 to $f : X \rightarrow C$, which implies $\mathcal{F} = \pi^*V' \otimes f^*L'$ for a stable bundle V' on B and a line bundle L' on C . Then $\deg \mathcal{F}_y = 2\deg L'$ which contradict to that \mathcal{F}_y is of degree 1.) Then, same as the proof of Proposition 4.3 of [6], we can prove that $\Delta(\mathcal{F}) \geq 10$. On the other hand, by (36), we have

$$\Delta(\mathcal{F}) = 4c_2(\mathcal{F}) - 2(1 - \deg f_*E) = 4c_2(\mathcal{F}) - 2$$

which implies $c_2(\mathcal{F}) \geq 3$ and

$$\Delta(E) = 6\deg f_*E + 6c_2(\mathcal{F}) - 4 = 6c_2(\mathcal{F}) - 4 \geq 14.$$

Now we assume that $\deg f_*E \leq -1$, which means that $\sum_{i=1}^s \text{rk} G_i \geq 4 + 3\dim H^0(G_s)$. When $\text{rk} G_s = 2$, if $\mu(G_s) < 0$, then $\Delta(E) \geq \text{Min}_{s_1+2s_2+2 \geq 4} \{2s_1 + 4s_2 + 6(\frac{1}{3} - (-\frac{1}{2})) \cdot 2\} \geq 14$ by (31); if $\mu(G_s) = 0$, tensoring E with a suitable line bundle π^*L^{-1} , we can assume that $H^0(G_s) \neq 0$ (cf. Theorem 5 of [11]), then $\Delta(E) \geq \text{Min}_{s_1+2s_2+2 \geq 7} \{2s_1 + 4s_2 + 6 \cdot \frac{1}{3} \cdot 2\} \geq 14$. Now we assume that $\text{rk} G_s = 1$, if $\mu(G_s) < 0$, then $\Delta(E) \geq \text{Min}_{s_1+2s_2+1 \geq 4} \{2s_1 + 4s_2 + 6(\frac{1}{3} - (-1))\} \geq 14$; if $\mu(G_s) = 0$, tensoring E with $\pi^*G_s^{-1}$, we can assume that $H^0(G_s) \neq 0$, then $\Delta(E) \geq \text{Min}_{s_1+2s_2+1 \geq 7} \{2s_1 + 4s_2 + 6 \cdot \frac{1}{3}\} \geq 14$.

If $\phi : B \rightarrow M$ passes through a generic point, we claim that $\deg f_*E \leq -2$, which implies $\Delta(E) \geq 20$. To prove the claim, we will prove that $\phi(B)$ lies in a proper closed subset if $\deg f_*E = -1$ or 0. If $\deg f_*E = -1$, note that \mathcal{F}_y must be locally free of degree 2 for generic $y \in B$ (if \mathcal{F}_y has nontrivial torsion, then E_y has a quotient bundle of rank 2 and degree at most 1, which is impossible since E_y is (1,1)-stable for generic $y \in B$). Thus E_y satisfies $0 \rightarrow V_1 \rightarrow E_y \rightarrow V_2 \rightarrow 0$ where V_1, V_2 are vector bundles on C of ranks 1, 2 and degrees -1, 2 respectively such that $V_1 \otimes \det V_2 = \mathcal{L}$. To estimate the dimension of the locus of such bundles, we can assume that both V_1 and V_2 is stable. The locus of such bundles has dimension at most $g + 4(g-1) + 1 + h^1(V_2^* \otimes V_1) - 1 - g = 6(g-1) + 4 < \dim M$ when $g \geq 4$. Similarly, if $\deg f_*E = 0$, we can show that $\phi(B)$ lies in a locus of dimension at most $6(g-1) + 1 < \dim M$. \square

Lemma 4.7. *If $g \geq 12$, M contains (1,10)-stable points.*

Proposition 4.8. *If E is semi-stable of degree 2 at the generic fiber of $f : X \rightarrow C$, $\Delta(E) \geq 8$. If $g > 12$ and $\phi : B \rightarrow M$ passes through the generic points, $\Delta(E) \geq 18$.*

Proof. It's known that there is a unique stable rank 3 vector bundle with a fixed determinant of degree 2 on an elliptic curve. Thus $\Delta(E) > 0$ if and only if there exists $t_1 \in C$ such that $E_{t_1} = E|_{X_{t_1}}$ is not semistable.

Let $E_{t_1} \rightarrow G_1 \rightarrow 0$ be a indecomposable quotient of minimal slop and

$$0 \rightarrow E^{(1)} \rightarrow E \rightarrow_{X_{t_1}} G_1 \rightarrow 0$$

be the elementary transformation of E along G_1 at X_{t_1} . If $E^{(i)}$ is defined and $\Delta(E^{(i)}) > 0$, let $t_{i+1} \in C$ such that $E_{t_{i+1}}^{(i)} = E^{(i)}|_{X_{t_{i+1}}}$ is not semi-stable and $E_{t_{i+1}}^{(i)} \rightarrow G_{i+1} \rightarrow 0$ be a indecomposable quotient of minimal slop, then we define $E^{(i+1)}$ to be the elementary transformation of $E^{(i)}$ along G_{i+1} at $X_{t_{i+1}}$, namely $E^{(i+1)}$ satisfies the exact sequence

$$0 \rightarrow E^{(i+1)} \rightarrow E^{(i)} \rightarrow_{X_{t_{i+1}}} G_{i+1} \rightarrow 0.$$

Let s be the minimal integer such that $\Delta(E^{(s)}) = 0$ and

$$\Delta(E) = \sum_{i=1}^s 6\left(\frac{2}{3} - \mu(G_i)\right)\text{rk} G_i, \quad (38)$$

where $\mu(G_i) \leq \frac{1}{2}$ if $\text{rk} G_i = 2$ and $\mu(G_i) \leq 0$ if $\text{rk} G_i = 1$. Let

$$s_1 = \#\{i : \text{rk} G_i = 1 \text{ but } i \neq s\} \text{ and } s_2 = \#\{i : \text{rk} G_i = 2 \text{ but } i \neq s\}.$$

Then

$$s_1 + s_2 + 1 = s \text{ and } s_1 + 2s_2 + \text{rk}G_s = \sum_{i=1}^s \text{rk}G_i,$$

and

$$\Delta(E) = \sum_{i=1}^s 6\left(\frac{2}{3} - \mu(G_i)\right) \text{rk}G_i \geq 4s_1 + 2s_2 + 6\left(\frac{2}{3} - \mu(G_s)\right) \text{rk}G_s, \quad (39)$$

Same as the above proposition, we have

$$\deg f_* E^{(s)} \leq \deg f_* E - \dim H^0(G_s) \quad (40)$$

and

$$\deg E^{(s)} = 1 - \sum_{i=1}^s \text{rk}G_i. \quad (41)$$

On the other hand, by Theorem 2.4, $\Delta(E^{(s)}) = 0$ implies that there is a stable vector bundle V of rank 3 and degree 2 on B and a line bundle L on C such that $E^{(s)} = \pi^*V \otimes f^*L$. It's easy to see

$$\deg E_y^{(s)} = 3 \deg L \quad (42)$$

and

$$\deg f_* E^{(s)} = 2 \deg L. \quad (43)$$

Thus

$$2 \sum_{i=1}^s \text{rk}(G_i) \geq 2 - 3 \deg f_* E + 3 \dim H^0(G_s). \quad (44)$$

We claim that $\deg f_* E \leq -1$. To show it, consider

$$0 \rightarrow \mathcal{F}' = f^* f_* E \rightarrow E \rightarrow \mathcal{F} \rightarrow 0$$

where \mathcal{F} is locally free of rank 1 on $f^{-1}(C \setminus T)$ and $T \subset C$ is a finite set such that $E_t (t \in T)$ is not semi-stable. Thus, for any $y \in B$, the sequence

$$0 \rightarrow \mathcal{F}'_y \rightarrow E_y \rightarrow \mathcal{F}_y \rightarrow 0 \quad (45)$$

is still exact, which implies that \mathcal{F} is B -flat (cf Lemma 2.1.4 of [7]). The sequence (45) already implies $\deg f_* E = \deg \mathcal{F}'_y \leq 0$ since E_y is stable of degree 1. Thus \mathcal{F} can not be locally free since

$$6c_2(\mathcal{F}) = \Delta(E) - 12 \deg f_* E + 8 > 0.$$

Then there is at least a $y_0 \in B$ such that \mathcal{F}_{y_0} has torsion, otherwise \mathcal{F} is locally free (cf Lemma 1.27 of [10]). The stability of E_{y_0} implies that $\mathcal{F}_{y_0}/\text{torsion}$ has degree at least 1. Thus $\deg \mathcal{F}_{y_0} \geq 2$ and

$$\deg f_* E = \deg \mathcal{F}'_{y_0} \leq -1.$$

Which means $2 \sum_{i=1}^s \text{rk}G_i \geq 5 + 3 \dim H^0(G_s)$. When $\text{rk}G_s = 1$, if $\mu(G_s) < 0$, then $\Delta(E) \geq \text{Min}_{2(s_1+2s_2+1) \geq 5} \{4s_1 + 2s_2 + 6(\frac{2}{3} - (-1))\} \geq 12$. If $\mu(G_s) = 0$, tensoring E with $\pi^*G_s^{-1}$, we can assume that $H^0(G_s) \neq 0$, then $\Delta(E) \geq \text{Min}_{2(s_1+2s_2+1) \geq 8} \{4s_1 + 2s_2 + 6 \cdot \frac{2}{3}\} \geq 8$. Now we consider the case $\text{rk}G_s = 2$. If $\mu(G_s) < 0$, then $\Delta(E) \geq \text{Min}_{2(s_1+2s_2+2) \geq 5} \{4s_1 + 2s_2 + 6(\frac{2}{3} - (-\frac{1}{2})) \cdot 2\} \geq 16$. If $\mu(G_s) = 0$, tensoring E with a suitable line bundle π^*L^{-1} , we can assume that $H^0(G_s) \neq 0$ (cf. Theorem 5 of [11]), then $\Delta(E) \geq \text{Min}_{2(s_1+2s_2+2) \geq 8} \{4s_1 + 2s_2 + 6 \cdot \frac{2}{3} \cdot 2\} \geq 8$. If $\mu(G_2) = \frac{1}{2}$, we can prove that $\Delta(E) = \sum_{i=1}^{s-1} 6(\frac{2}{3} - \mu(G_i)) \text{rk}G_i + 2 \geq 8$ as following:

If $s_1 > 1$, it's easy to see that $\Delta(E) \geq 4s_1 + 2s_2 + 2 \geq 10$. If $s_1 = 1$ then $s_2 \geq 1$ since $2(s_1 + 2s_2 + 2) \geq 2 - 3 \deg f_* E + 3 \dim H^0(G_s) = 5 - 3 \deg f_* E \geq 8$, thus $\Delta(E) \geq 4s_1 + 2s_1 + 2 \geq 4 \times 1 + 2 \times 1 + 2 = 8$. Now we assume that $s_1 = 0$, then we must have either $s_2 > 2$ or $\exists i$ such that $\mu(G_i) \leq 0$. (In fact, we note that $s_2 \geq 1$ since $2(s_1 + 2s_2 + 2) \geq 2 - 3 \deg f_* E + 3 \dim H^0(G_s) = 5 - 3 \deg f_* E \geq 8$, if $s_2 = 1$ and $\mu(G_1) = \mu(G_2) = \frac{1}{2}$, then $s = 2$ and, by the following Lemma, we

have $\deg f_* E^{(2)} = \deg f_* E - 2 = -3$ since $2(s_1 + 2s_2 + 2) \geq 5 - 3\deg f_* E \geq 8$, which contradicts to equation (43). If $s_2 = 2$ and $\mu(G_1) = \mu(G_2) = \mu_{G_3} = \frac{1}{2}$, then $s = 3$ and $\deg E_y^{(3)} = 1 - \sum_{i=1}^3 2 = -5$ by (41), which contradicts equation (42). If $s_2 > 2$, we also have $\Delta(E) \geq 2s_2 + 2 \geq 8$. If $\exists i$ such that $\mu(G_i) \leq 0$, then $\Delta(E) = \sum_{i=1}^{s-1} 6(\frac{2}{3} - \mu(G_i))\text{rk}G_i + 2 \geq 6 \times \frac{2}{3} \times 2 + 2 = 10$.

If $\phi : B \rightarrow M$ passes through a generic point, i.e., a $(1, 10)$ -stable point, we claim that $\deg f_* E \leq -8$. To prove the claim, we assume that $\deg f_* E = -m$ (where $m \in \{1, 2, 3, 4, 5, 6, 7\}$), we will show that $\phi(B)$ lies in a proper closed subset. Note that \mathcal{F}_y must be locally free of rank 1 and degree $1 + m$ for generic $y \in B$ (if \mathcal{F}_y has nontrivial torsion, then E_y has a quotient line bundle of degree at most m , which is impossible since E_y is $(1, 10)$ -stable for generic $y \in B$). Thus E_y satisfies $0 \rightarrow V_1 \rightarrow E_y \rightarrow V_2 \rightarrow 0$, where V_1, V_2 are vector bundles on C of ranks 2, 1 and degrees $-m, 1 + m$ respectively such that $\det V_1 \times V_2 \cong \mathcal{L}$. The locus of such bundles has dimension at most $4(g-1) + 1 + g + h^1(V_2^{-1} \otimes V_1) - 1 - g = 6(g-1) + 3m + 2 < \dim M$ when $g > 12$. Thus $\deg f_* E \leq -8$ and $2 \sum_{i=1}^s \text{rk}(G_i) \geq 2 - 3\deg f_* E + 3\dim H^0(G_s) \geq 26 + 3\dim H^0(G_s)$. We consider the case that $\text{rk}(G_s) = 1$ at first, if $\mu(G_s) < 0$, then $\Delta(E) \geq \text{Min}_{2(s_1+2s_2+1) \geq 26} \{4s_1 + 2s_2 + 6(\frac{2}{3} - (-1))\} \geq 22$. If $\mu(G_s) = 0$, tensoring E with $\pi^* G_s^{-1}$, we can assume that $H^0(G_s) \neq 0$, then $\Delta(E) \geq \text{Min}_{2(s_1+2s_2+1) \geq 29} \{4s_1 + 2s_2 + 6 \times \frac{2}{3}\} \geq 18$. Now we consider the case that $\text{rk}(G_s) = 2$. If $\mu(G_s) < 0$, then $\Delta(E) \geq \text{Min}_{2(s_1+2s_2+2) \geq 26} \{4s_1 + 2s_2 + 6(\frac{2}{3} - (-\frac{1}{2})) \cdot 2\} \geq 22$. If $\mu(G_s) = 0$, tensoring E with a suitable line bundle $\pi^* L^{-1}$, we can assume that $H^0(G_s) \neq 0$ (cf. Theorem 5 of [11]), then $\Delta(E) \geq \text{Min}_{2(s_1+2s_2+2) \geq 29} \{4s_1 + 2s_2 + 6 \times \frac{2}{3} \times 2\} \geq 22$. If $\mu(G_s) = \frac{1}{2}$, if $s_1 \geq 1$, then $\Delta(E) \geq 2s_2 + 6(\frac{2}{3} - \frac{1}{2}) \times 2 \geq 18$ since $2(s_1 + 2s_2 + 2) \geq 26 + 3 = 29$. If $s_1 = 0$, considering the inequality (44), then $\Delta(E) = \sum_{i=1}^{s-1} 6(\frac{2}{3} - \frac{1}{2}) \times 2 < 18$ if and only if $s_2 = 7$, $\mu(G_1) = \dots = \mu(G_7) = \mu(G_8) = \frac{1}{2}$ and $-9 \leq \deg f_* E \leq -8$. By the following Lemma, we have $\deg f_* E^{(s)} = \deg f_* E - 8$. By equation (43), $\deg f_* E = -8$ and $\deg L = -8$. But, on the other hand, by equations (41) and (42), we have $\deg L = -5$. The contradiction implies that $\Delta(E) \geq 18$. \square

Lemma 4.9. *Keep the notations as Proposition 4.8. If for any $i \in \{1, \dots, s\}$, $\text{rk}G_i = 2$ and $\mu(G_i) = \frac{1}{2}$, then we have $\deg f_* E^{(s)} = \deg f_* E - s$.*

Proof. Since G_i is an indecomposable vector bundle on $X_{t_i} = \{t_i\} \times B \cong B$ of rank 2 and degree 1, then by Lemma 15 in [11] and Riemann-Roch Theorem, we have

$$\dim H^0(G_i) = 1 \quad \text{and} \quad H^1(G_i) = 0.$$

By the definition of $E^{(i)}$, we have

$$0 \rightarrow E^{(i)} \rightarrow E^{(i-1)} \rightarrow_{X_{t_i}} G_i \rightarrow 0.$$

Take direction of above sequence, we have

$$0 \rightarrow f_* E^{(i)} \rightarrow f_* E^{(i-1)} \rightarrow H^0(G_i) \rightarrow R^1 f_* E^{(i)} \rightarrow R^1 f_* E^{(i-1)} \rightarrow 0. \quad (46)$$

If $R^1 f_* E^{(i)} = 0$, then by (46), we have

$$\deg f_* E^{(i)} = \deg f_* E^{(i-1)} - 1 \quad \text{and} \quad R^1 f_* E^{(i-1)} = 0. \quad (47)$$

We note that $R^1 f_* E^{(s)} = 0$, then $R^1 f_* E^{(s-1)} = \dots = R^1 f_* E^{(1)} = R^1 f_* E = 0$ and

$$\deg f_* E^{(s)} = \deg f_* E^{(s-1)} - 1 = \dots = \deg f_* E - s.$$

\square

Before consider the case that E is semi-stable of degree 0 on the generic fiber of $f : X \rightarrow C$, we note that:

(1) For any vector bundle E , $\Delta(E) = \Delta(E^*)$ where E^* is the dual of E .

(2) If $\phi : B \rightarrow M = SU_C(3, \mathcal{L})$ is defined by a vector bundle E on $C \times B$, let $\phi_{E^*} : B \rightarrow M^* = SU_C(3, \mathcal{L}^{-1})$ be the morphism defined by E^* . Then $\phi : B \rightarrow M = SU_C(3, \mathcal{L})$ can factors as the composition of ϕ_{E^*} with the natural isomorphism $M^* \cong M, W \mapsto W^*$.

(3) $E_t^* = E^*|_{X_t}$ is semi-stable on a fiber $X_t = f^{-1}(t)$ if and only if E_t is semi-stable.

Now we consider the case that E is semi-stable of degree 0 on the generic fiber of $f : X \rightarrow C$. If E is semi-stable on every fiber of $f : X \rightarrow C$, then E induces a non-trivial morphism

$$\varphi_E : C \rightarrow \mathbb{P}^2$$

(cf. [12]) such that $\varphi_E^* \mathcal{O}_{\mathbb{P}^2}(1) = (\det f_! E)^{-1}$, which has degree $c_2(E)$ by Grothendieck-Riemann-Roch theorem. Thus

$$\Delta(E) = 6c_2(E) = 6\deg \varphi_E \geq 12. \quad (48)$$

If there is a $t_0 \in C$ such that $E_{t_0} = E|_{X_{t_0}}$ is not semi-stable on $X_{t_0} = f^{-1}(t_0)$, let $E_{t_0} \rightarrow G \rightarrow 0$ be the indecomposable quotient bundle of minimal slope μ . If $\text{rk} G = 1$, G is a line bundle of degree $\mu = \deg G$ and we will denote $\mathcal{O}(\mu) := G$, and $E' = \ker(E \rightarrow_{X_{t_0}} \mathcal{O}(\mu) \rightarrow 0)$. If $\text{rk} G = 2$, let $\mu^* := 2\mu = \deg G$. Then $\ker(E_{t_0} \rightarrow G \rightarrow 0)$ is a line bundle of degree $-\mu^*$. Take dual, $E_{t_0}^*$ is not semi-stable and $E_{t_0}^* \rightarrow \mathcal{O}(\mu^*) \rightarrow 0$ is a minimal quotient line bundle of degree μ^* . Let $E^{*\prime} = \ker(E^* \rightarrow_{X_{t_0}} \mathcal{O}(\mu^*) \rightarrow 0)$.

Lemma 4.10. *If $\text{rk} G = 1$ and $\Delta(E') = 0$, then there is a semi-stable vector bundle V on C and a line bundle L of degree 0 on B such that*

$$E' = f^* V \otimes \pi^* L.$$

Proof. By the definition, $\{E'_t := E'|_{\{t\} \times B}\}_{t \in C}$ and $\{E'_y := E'|_{C \times \{y\}}\}_{y \in B}$ are families of semi-stable bundles of degree 0. Apply Theorem 2.4 to $f : X \rightarrow C$ (resp. $\pi : X \rightarrow B$), then $\Delta(E') = 0$ implies that $\{E'_y\}_{y \in B}$ (resp. $\{E'_t\}_{t \in C}$) are isomorphic to each other. Let L be the line bundle of degree 0 on B such that $H^0(E'_t \otimes L^{-1})(\forall t \in C)$ have maximal dimension. By tensoring E (thus E') with $\pi^* L^{-1}$, we can assume that $H^0(E'_t) \neq 0(\forall t \in C)$, which have dimension at most 3 since E'_t is semi-stable of degree 0. If $H^0(E'_t)$ has dimension 3, then $E' = f^*(f_* E')$ and we are done. If $H^0(E'_t)$ has dimension 1 or 2, we will show contradictions.

By the definition of E' , we have an exact sequence

$$0 \rightarrow E' \rightarrow E \rightarrow_{X_{t_0}} \mathcal{O}(\mu) \rightarrow 0, \quad (49)$$

where $\mathcal{O}(\mu)$ is a line bundle of degree $\mu < 0$ on B . Then

$$V_1 := f_* E = f_* E'$$

is a vector bundle on C .

If $H^0(E'_t)$ has dimension 1, then $V_1 := f_* E' = f_* E$ is a line bundle and we have exact sequence

$$0 \rightarrow f^* V_1 \rightarrow E' \rightarrow \mathcal{F}' \rightarrow 0 \quad (50)$$

for a rank 2 vector bundle \mathcal{F}' on $C \times B$ and $\Delta(\mathcal{F}') = 0$. All the bundles $\{\mathcal{F}'_t = \mathcal{F}'|_{\{t\} \times B}\}_{t \in C}$ are semi-stable of degree 0 and are isomorphic to each other since all the bundles $\{E'_t\}_{t \in C}$ are semi-stable of degree 0 and are isomorphic to each other. Let L' be a line bundle of degree 0 on B such that $H^0(\mathcal{F}'_t \otimes L') \neq 0(\forall t \in C)$, which must have dimension 1. To see it, tensor (50) with $\pi^* L'$, we have an exact sequence

$$0 \rightarrow f^* V_1 \otimes \pi^* L' \rightarrow E' \otimes \pi^* L' \rightarrow \mathcal{F}' \otimes \pi^* L' \rightarrow 0. \quad (51)$$

Restrict (51) to $X_t = \{t\} \times B \cong B$, we have

$$0 \rightarrow L' \rightarrow E'_t \otimes L' \rightarrow \mathcal{F}'_t \otimes L' \rightarrow 0,$$

and then we have

$$0 \rightarrow H^0(L') \rightarrow H^0(E'_t \otimes L') \rightarrow H^0(\mathcal{F}'_t \otimes L') \rightarrow H^1(L') \rightarrow \cdots,$$

which implies $h^0(\mathcal{F}'_t \otimes L') \leq h^0(E'_t \otimes L') \leq 1$ by the choice of L . Then $V_2 = f_*(\mathcal{F}' \otimes \pi^* L')$ is a line bundle on C , and we have exact sequence

$$0 \rightarrow f^* V_2 \rightarrow \mathcal{F}' \otimes \pi^* L' \rightarrow f^* V_3 \otimes \pi^* L'' \rightarrow 0 \quad (52)$$

for a line bundle V_3 on C and a degree 0 line bundle L'' on B .

Now we note that $\deg V_1 + \deg V_2 \leq -1$. To see it, we consider the exact sequence

$$0 \rightarrow f^* V_1 \rightarrow E \rightarrow \mathcal{F} \rightarrow 0 \quad (53)$$

where $\mathcal{F}|_{f^{-1}(C \setminus \{t_0\})}$ is locally free and \mathcal{F} satisfies

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow_{X_{t_0}} \mathcal{O}(\mu) \rightarrow 0. \quad (54)$$

Thus $f_*(\mathcal{F} \otimes \pi^* L') = f_*(\mathcal{F}' \otimes \pi^* L') = V_2$ since $\mu < 0$. Then we have an exact sequence

$$0 \rightarrow f^* V_2 \rightarrow \mathcal{F} \otimes \pi^* L' \rightarrow \mathcal{G} \rightarrow 0 \quad (55)$$

where $\mathcal{G}|_{f^{-1}(C \setminus \{t_0\})}$ is locally free of rank 1 by (52). But \mathcal{G} is not locally free (otherwise $c_2(E) = c_2(E \otimes \pi^* L') = c_1(f^* V_1 \otimes \pi^* L')(c_1(E \otimes \pi^* L') - c_1(f^* V_1 \otimes \pi^* L')) + c_2(\mathcal{F} \otimes \pi^* L') = c_1(f^* V_2)(c_1(E \otimes \pi^* L') - c_1(f^* V_1 \otimes \pi^* L') - c_1(f^* V_2)) = 0$), and for any $y \in B$, the restrictions of (53) and (55) to $X_y = \pi^{-1}(y)$

$$0 \rightarrow V_1 \rightarrow E_y \rightarrow \mathcal{F}_y \rightarrow 0 \text{ and } 0 \rightarrow V_2 \rightarrow \mathcal{F}_y \rightarrow \mathcal{G}_y \rightarrow 0$$

are still exact, which means \mathcal{F} is B -flat and then \mathcal{G} is B -flat (cf. Lemma 2.1.4 of [7]). Thus, by Lemma 1.27 of [10], there is a $y_0 \in B$ such that \mathcal{G}_{y_0} has torsion $\tau \neq 0$ since \mathcal{G} is not locally free. Then, since E_{y_0} is stable of degree 1,

$$\deg \mathcal{G}_{y_0} \geq 1 + \deg \frac{\mathcal{G}_{y_0}}{\tau} > 1 + \mu(E_{y_0}) = \frac{4}{3}$$

which implies $\deg V_1 + \deg V_2 = \deg E_{y_0} - \deg \mathcal{G}_{y_0} \leq -1$.

By the sequences (51) and (52), $f^* V_3 \otimes \pi^* L''$ is a quotient line bundle of $E' \otimes \pi^* L'$. Let $F := \ker(E' \otimes \pi^* L' \rightarrow f^* V_3 \otimes \pi^* L'')$, then there is an induced morphism $\lambda : F \rightarrow f^* V_2$ satisfying the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F & \longrightarrow & E' \otimes \pi^* L' & \longrightarrow & f^* V_3 \otimes \pi^* L'' \longrightarrow 0 \\ & & \lambda \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & f^* V_2 & \longrightarrow & \mathcal{F}' \otimes \pi^* L' & \longrightarrow & f^* V_3 \otimes \pi^* L'' \longrightarrow 0. \end{array}$$

By the snake lemma, λ is surjective and $\ker \lambda = f^* V_1 \otimes \pi^* L'$. Then F fits an exact sequence

$$0 \rightarrow f^* V_1 \otimes \pi^* L' \rightarrow F \rightarrow f^* V_2 \rightarrow 0, \quad (56)$$

which is determined by a class in $H^1(X, f^*(V_2^{-1} \otimes V_1) \otimes \pi^* L')$.

If $L' \neq \mathcal{O}_B$, then $R^i f_*(f^*(V_2^{-1} \otimes V_1) \otimes \pi^* L') = V_2^{-1} \otimes V_1 \otimes H^i(L') = 0 (i = 0, 1)$, which implies $H^1(X, f^*(V_2^{-1} \otimes V_1) \otimes \pi^* L') = 0$ and (56) is split. Thus there is a section $f^* V_2 \rightarrow F$ of λ , and we can consider $f^* V_2$ as a sub line bundle of $E' \otimes \pi^* L'$ by the morphism $f^* V_2 \rightarrow F \rightarrow E' \otimes \pi^* L'$. Then $\deg V_2 \leq 0$ since E'_y is semi-stable of degree 0 for any $y \in B$. If $L'' \neq \mathcal{O}_B$, then (52) is also split, then we have an exact sequence of inverse direction

$$0 \rightarrow f^* V_3 \otimes \pi^* L'' \rightarrow \mathcal{F}' \otimes \pi^* L' \rightarrow f^* V_2 \rightarrow 0.$$

Hence $\deg V_2 = 0$. Now we let $F' = \ker(E' \otimes \pi^* L' \rightarrow f^* V_2 \rightarrow 0)$, then there is an induced morphism $F' \rightarrow f^* V_3 \otimes \pi^* L''$ and F' satisfies an exact sequence

$$0 \rightarrow f^* V_1 \otimes \pi^* L' \rightarrow F' \rightarrow f^* V_3 \otimes \pi^* L'' \rightarrow 0, \quad (57)$$

which is determined by a class in $H^1(X, f^*(V_3^{-1} \otimes V_1) \otimes \pi^*(L''^{-1} \otimes L'))$. If $L'' \neq L'$, we can prove that (57) is split. Then $f^*V_3 \otimes \pi^*L''$ is a sub line bundle of F' and then $f^*V_3 \otimes \pi^*L''$ is a sub line bundle of $E' \otimes \pi^*L'$. Thus $\deg V_3 \leq 0$ since E'_y is semi-stable of degree 0 for any $y \in B$, which contradicts that $\deg V_1 + \deg V_2 \leq -1$. Hence $L'' = L'$. Tensoring (57) with $\pi^*L'^{-1}$, we have an exact sequence

$$0 \rightarrow f^*V_1 \rightarrow F' \otimes \pi^*L'^{-1} \rightarrow f^*V_3 \rightarrow 0, \quad (58)$$

which is determined by a class in $H^1(X, f^*(V_3^{-1} \otimes V_1))$. However, note $R^if_*(f^*(V_3^{-1} \otimes V_1)) = V_3^{-1} \otimes V_1 (i = 0, 1)$ and $H^0(C, V_3^{-1} \otimes V_1) = 0$ since $\deg V_2 = 0$, by Leray spectral sequence, we have

$$H^1(C, V_3^{-1} \otimes V_1) \cong H^1(X, f^*(V_3^{-1} \otimes V_1)).$$

Hence there exists an extension $0 \rightarrow V_1 \rightarrow V' \rightarrow V_3 \rightarrow 0$ on C such that $F' \otimes \pi^*L'^{-1} = f^*V'$. Thus $h^0(E'_t) \geq h^0((F' \otimes \pi^*L'^{-1})_t) = 2$, which contradicts the assumption $h^0(E'_t) = 1$. Thus L'' has to be \mathcal{O}_B and (52) has to be

$$0 \rightarrow f^*V_2 \rightarrow \mathcal{F}' \otimes \pi^*L' \rightarrow f^*V_3 \rightarrow 0 \quad (59)$$

which is determined by a class in $H^1(X, f^*(V_3^{-1} \otimes V_2))$. However, note that $R^if_*(f^*(V_3^{-1} \otimes V_2)) = V_3^{-1} \otimes V_2 (i = 0, 1)$ and $H^0(C, V_3^{-1} \otimes V_2) = 0$ since $\deg V_2 \leq 0$, by Leray spectral sequence, we have

$$H^1(C, V_3^{-1} \otimes V_2) \cong H^1(X, f^*(V_3^{-1} \otimes V_2)).$$

Hence there exists an extension $0 \rightarrow V_2 \rightarrow W' \rightarrow V_3 \rightarrow 0$ on C such that $\mathcal{F}' \otimes \pi^*L' = f^*W'$. Thus $h^0(E'_t \otimes L') \geq h^0((\mathcal{F}' \otimes \pi^*L')_t) = 2$, which contradict the choice of L and that $h^0(E'_t) = 1$.

We have shown that L' has to be \mathcal{O}_B and (56) has to be

$$0 \rightarrow f^*V_1 \rightarrow F \rightarrow f^*V_2 \rightarrow 0, \quad (60)$$

which is determined by a class in $H^1(X, f^*(V_2^{-1} \otimes V_1))$. If $\deg V_2 > \deg V_1$, we can see that

$$H^1(C, V_2^{-1} \otimes V_1) \cong H^1(X, f^*(V_2^{-1} \otimes V_1)).$$

Hence there exists an extension $0 \rightarrow V_1 \rightarrow W \rightarrow V_2 \rightarrow 0$ on C such that $F = f^*W$. Thus $h^0(E'_t) \geq h^0(F_t) = 2$, which contradicts that $h^0(E'_t) = 1$. So $\deg V_2 \leq \deg V_1$. Since $\deg V_1 + \deg V_2 \leq -1$ and $\deg V_1 \leq 0$, then $\deg V_2 \leq -1$ and $L'' = \mathcal{O}_B$ (otherwise, the sequence (52) is split and then $\deg V_2 \geq 0$). Now (52) has to be (59), which is determined by a class in $H^1(X, f^*(V_3^{-1} \otimes V_2)) \cong H^1(C, V_3^{-1} \otimes V_2)$. which implies that $h^0(E'_t \otimes L') \geq h^0((\mathcal{F}' \otimes \pi^*L')_t) = 2$, which contradict the choice of L and that $h^0(E'_t) = 1$.

If $H^0(E'_t)$ has dimension 2, we can also show a contradiction similar as in Lemma 4.4 in [6]. \square

Proposition 4.11. *When E is semi-stable of degree 0 on the generic fiber of $f : X \rightarrow C$, we have $\Delta(E) \geq 6$. If C is not hyper-elliptic and $\phi : B \rightarrow M$ passes through generic points, assume that E defines an essential elliptic curves, then $\Delta(E) \geq 18$ when $g \geq 4$.*

Proof. If E is semi-stable on each fiber of $f : X \rightarrow C$, then E induces a non-trivial morphism $\varphi_E : B \rightarrow \mathbb{P}^2$. By (48), $\Delta(E) \geq 12$.

If there is a $t_0 \in C$ such that E_{t_0} is not semi-stable, then we have either (49) or

$$0 \rightarrow E^{*'} \rightarrow E^* \rightarrow_{X_{t_0}} \mathcal{O}(\mu^*) \rightarrow 0, \quad (61)$$

where $\mathcal{O}(\mu^*)$ is a line bundle of degree $\mu^* \leq -1$ on B .

If we have (49). If $\Delta(E') \neq 0$, then $\Delta(E') > 0$ by Theorem 2.4. On the other hand, $c_1(E')^2 = 0$ since E' has degree 0 on the generic fiber of $f : X \rightarrow C$ and $\text{Pic}(C \times B) = \text{Pic}(C) \times \text{Pic}(B)$. Thus $\Delta(E') = 6c_2(E') \geq 6$, and by Lemma 2.5, we have $\Delta(E) = \Delta(E') - 6\mu \geq 12$. If $\Delta(E') = 0$, by Lemma 4.10, we can assume that $E' = f^*V$, then the sequence (49) induces a nontrivial morphism $\varphi : B \rightarrow \mathbb{P}(V_{t_0}^*)$ such that $\mathcal{O}(-\mu) = \varphi^* \mathcal{O}_{\mathbb{P}(V_{t_0}^*)}(1)$. Thus $\Delta(E) = -6\mu \geq 12$.

If we have (61), by Lemma 2.5, we have $\Delta(E) = \Delta(E^*) = \Delta(E^{*'}) - 6\mu^* \geq 6$.

Now we assume that C is not hyper-elliptic and $\phi : B \rightarrow M$ passes through generic points, i.e., $E_y|_{C \times \{y\}}$ is not only (1,1)-stable but also (1,2)-stable for generic $y \in B$. If E is semi-stable on each fiber X_t , then $\Delta(E) \geq 6\deg\varphi_E \geq 18$ by (48) since C is not hyper-elliptic.

If there is a $t_0 \in C$ such that E_{t_0} is not semi-stable, then we have either (49) or (61).

We consider the case that (49) holds at first, for this case we have $\Delta(E) = \Delta(E') - 6\mu$. If $\Delta(E') = 0$, then $E' = f^*V$ where V is a (1,0)-stable by Lemma 3.3, then the sequence (49) induces a non-trivial morphism $\varphi : B \rightarrow \mathbb{P}(V_{t_0}^*)$ such that $\mathcal{O}(-\mu) = \varphi^*\mathcal{O}_{\mathbb{P}(V_{t_0}^*)}(1)$ and $\phi : B \rightarrow M$ factors through $\varphi : B \rightarrow \varphi(B) \subset \mathbb{P}(V_{t_0}^*)$, which implies that the normalization of $\varphi(B)$ is an elliptic curve. Hence $-\mu \geq 3$ and $\Delta(E) \geq 18$.

Now we consider the case $\Delta(E') > 0$. We claim that $\Delta(E') \geq 12$, which implies $\Delta(E) \geq 18$. If E' is semi-stable on each fiber X_t , then E' defines a non-trivial morphism $\varphi_{E'} : C \rightarrow \mathbb{P}^2$ such that $\varphi_{E'}^*\mathcal{O}_{\mathbb{P}^2}(1) = (\det f_!E')^{-1} = c_2(E')$. Thus $\Delta(E') \geq 12$. If there is a $t'_0 \in C$ such that $E'_{t'_0}$ is not semi-stable, then we have either

$$0 \rightarrow E'' \rightarrow E' \rightarrow_{X_{t'_0}} \mathcal{O}(\mu') \rightarrow 0 \quad (62)$$

where $E''_y = E''|_{C \times \{y\}}$ is stable of degree -1 for generic $y \in B$ since E'_y is stable of degree 0, or

$$0 \rightarrow E'^{*'} \rightarrow E'^* \rightarrow_{X_{t'_0}} \mathcal{O}(\mu'^*) \rightarrow 0 \quad (63)$$

where $E'^{*'}_y = E'^{*'}|_{C \times \{y\}}$ is stable of degree -1 since $E'^*_y = (E'_y)^*$ is stable of degree 0. Suppose that (62) holds, if $\Delta(E'') \neq 0$, it's clear that $\Delta(E'') = 6c_2(E'') \geq 6$ and $\Delta(E') = \Delta(E'') - 6\mu' \geq 12$. If $\Delta(E'') = 0$, by Theorem 2.4, there is a stable bundle V' on C such that $E''_y = V'$ for all $y \in B$. Then we can choose $E'' = f^*V'$, the sequence (62) induces a non-trivial morphism $\varphi' : B \rightarrow \mathbb{P}(V'_{t'_0})$ such that $\mathcal{O}(-\mu') = \varphi'^*\mathcal{O}_{\mathbb{P}(V'_{t'_0})}(1)$. Thus $\Delta(E') = -6\mu' \geq 12$. Now we suppose (63) holds. If $\Delta(E'^{*'}) \neq 0$, it's clear that $\Delta(E'^{*'}) = 6c_2(E'^{*'}) \geq 6$ and $\Delta(E') = \Delta(E'^*) = \Delta(E'^{*'}) - 6\mu'^* \geq 12$. If $\Delta(E'^{*'}) = 0$, by Theorem 2.4, there is a stable bundle W' on C such that $E'^{*'}_y = W'$ for all $y \in B$. Then we can choose $E'^{*'} = f^*W'$, the sequence (63) induces a non-trivial morphism $\psi' : B \rightarrow \mathbb{P}(W'_{t'_0})$ such that $\mathcal{O}(-\mu'^*) = \psi'^*\mathcal{O}_{\mathbb{P}(W'_{t'_0})}(1)$. Thus $-\mu'^* \geq 2$ and $\Delta(E') = \Delta(E'^*) = -6\mu'^* \geq 12$.

For the case that (61), we have $\Delta(E) = \Delta(E^*) = \Delta(E'^*) - 6\mu^*$. If $\Delta(E'^*) = 0$, then $E'^* = f^{-1}W$ where W is (2,0)-stable of degree -2 by Remark 3.1(ii) and Lemma 3.3, then the sequence (61) induces a non-trivial morphism $\psi : B \rightarrow \mathbb{P}(W_{t_0}^*)$ such that $\mathcal{O}(-\mu^*) = \psi^*\mathcal{O}_{\mathbb{P}(W_{t_0}^*)}(1)$ and $\phi : B \rightarrow M$ factors through $\psi : B \rightarrow \psi(B) \subset \mathbb{P}(W_{t_0}^*)$ by $\mathbb{P}(W_{t_0}^*) \rightarrow M^* \cong M$. Which implies that the normalization of $\psi(B)$ is an elliptic curve. Hence $-\mu^* \geq 3$ and $\Delta(E) = \Delta(E^*) \geq 18$. For the case $\Delta(E'^*) \neq 0$, we can prove that $\Delta(E'^*) \geq 12$ similarly as to prove that $\Delta(E') \geq 12$, and hence $\Delta(E) = \Delta(E^*) \geq 18$. \square

From the Example 3.6 of [6] and Proposition 3.6, we can see the existence of essential elliptic curves of degree $6(r, d)$ (which is 6 in our case). By Propositions 4.1, 4.3, 4.5, 4.6, 4.8, 4.11, we have

Theorem 4.12. *Let $M = SU_C(3, \mathcal{L})$ be the moduli space of rank 3 stable bundles on C with fixed determinant of degree 1. Then, when C is generic, any essential elliptic curve $\phi : B \rightarrow M$ has degree*

$$\deg\phi^*(-K_M) \geq 6$$

and $\deg\phi^*(-K_M) = 6$ if and only if ϕ satisfies one of the following conditions:

(1) it factors through

$$\phi : B \xrightarrow{\psi} q^{-1}(\xi) = \mathbb{P}(H^1(V_2^* \otimes V_1)) \xrightarrow{\Phi_\xi} M$$

for some $\xi = (V_1, V_2) \in J_C \times U_C(2, 1)$ such that $\psi^*\mathcal{O}_{\mathbb{P}(H^1(V_2^* \otimes V_1))}(1)$ has degree 3.

(2) it factors through

$$\phi : B \xrightarrow{\psi} \mathcal{P} \xrightarrow{\Phi_\xi} M$$

but it is not in any fiber of $q : \mathcal{P} \rightarrow \mathcal{R}_\mathcal{L} \hookrightarrow J_C \times U_C(2, 1)$, and q induces a morphism $q_2 : B \rightarrow \mathbb{P}(H^1(L_3^{-1} \otimes L_2))$ for some $(L_2, L_3) \in J_C \times J_C^1$ such that $q_2^* \mathcal{O}_{\mathbb{P}(H^1(L_3^{-1} \otimes L_2))}(1)$ has degree 2 and $\psi^* \mathcal{O}_\mathcal{P}(1)$ has degree 1.

(3) it's defined by a vector bundle E on $C \times B$, which is semi-stable of degree 0 at generic fiber of $f : X \rightarrow C$, there exists only one point $t_0 \in C$ such that E_{t_0} is not semi-stable and the minimal slop indecomposable quotient bundle G of rank 2 and $\deg G = -1$.

Proof. By Propositions 4.1, 4.3, 4.5, 4.6, 4.8, 4.11, we have $\Delta(E) \geq 6$. Let $0 = E_0 \subset E_1 \subset \dots \subset E_n = E$ be the relative Harder-Narasimhan filtration of E over C . The possible case $\Delta(E) = 6$ only set up only in following three cases:

In Proposition 4.5 when $\deg E_1 = 0$, $\Delta(F_2) = 0$ and F_2 is semistable of even degree $2\mu_2$ at the generic fiber of $f : X \rightarrow C$. The condition $\deg E_1 = 0$ implies there is a line bundle V_1 of degree 0 on C and a line bundle $\mathcal{O}(\mu_1)$ of degree μ_1 on B such that $E_1 = f^* V_1 \otimes \pi^* \mathcal{O}(\mu_1)$. Since E is stable of degree 1 at every fiber of $\pi : X \rightarrow B$, F_2 is also stable of degree 1 at every fiber of $\pi : X \rightarrow B$. Applying Theorem 2.4 to $\pi : X \rightarrow B$, there is a stable bundle V_2 of degree 1 on C and a line bundle $\mathcal{O}(\mu_2)$ of degree μ_2 on B such that $F_2 = f^* V_2 \otimes \pi^* \mathcal{O}(\mu_2)$. These imply that $E \otimes \pi^* \mathcal{O}(-\mu_2)$ fits an exact sequence

$$0 \rightarrow f^* V_1 \otimes \pi^* \mathcal{O}(\mu_1 - \mu_2) \rightarrow E \otimes \pi^* \mathcal{O}(-\mu_2) \rightarrow f^* V_2 \rightarrow 0,$$

which defines a morphism $\phi : B \rightarrow \mathbb{P}(H^1(V_2^* \otimes V_1))$ such that $\psi^* \mathcal{O}_{\mathbb{P}(H^1(V_2^* \otimes V_1))}(1)$ is of degree $\mu_1 - \mu_2$. Then $\Delta(E) = 6$ and (19) imply $\mu_1 - \mu_2 = 3$.

In Proposition 4.3 when $c_2(F_2) = 0$, $\Delta(E) = 0$ and E_1 is semi-stable of odd degree $2\mu_1$ at generic fiber of $f : X \rightarrow C$. Tensoring E with $\pi^* \mathcal{O}(m)$ for a degree m line bundle $\mathcal{O}(m)$ on B such that $\deg(E_1 \otimes \pi^* \mathcal{O}(m)) = 2\mu_1 + 2m = 1$. Applying Theorem 2.4 to $f : X \rightarrow C$, $\Delta(E) = 0$ implies all the bundles $\{E_{1t} \otimes \mathcal{O}(m)\}_{t \in C}$ are semistable of degree 1 and s -equivalent each other, and then stable and isomorphic to each other. Thus there is a stable bundle V of degree 1 on B and a line bundle L_1 on C such that $E_1 \otimes \pi^* \mathcal{O}(m) = \pi^* V \otimes f^* L_1$. Then we have $\deg E_1 = 2\deg L_1$ and $f_*(E_1 \otimes \pi^* \mathcal{O}(m)) \cong L_1$. Thus $E_1 \otimes \pi^* \mathcal{O}(m)$ satisfies an exact sequence

$$0 \rightarrow f^* L_1 \rightarrow E_1 \otimes \pi^* \mathcal{O}(m) \rightarrow f^* L_2 \otimes \pi^* \mathcal{O}(1) \rightarrow 0,$$

for a line bundle L_2 on C and a line bundle $\mathcal{O}(1)$ of degree 1 on B . On the other hand, $c_2(F_2) = 0$ implies there is a line bundle L_3 on C and a line bundle $\mathcal{O}(\mu_2)$ of degree μ_2 on B such that $F_2 = f^* L_3 \otimes \pi^* \mathcal{O}(\mu_2)$. Then $E \otimes \pi^* \mathcal{O}(m)$ fits an exact sequence

$$0 \rightarrow E_1 \otimes \pi^* \mathcal{O}(m) \rightarrow E \otimes \pi^* \mathcal{O}(m) \rightarrow f^* L_3 \otimes \pi^* \mathcal{O}(\mu_2 + m) \rightarrow 0.$$

Consider $f^* L_1 \otimes \pi^* \mathcal{O}(-\mu_2 - m)$ as a subline bundle of $E \otimes \pi^* \mathcal{O}(-\mu_2)$ and let $E' := \frac{E \otimes \pi^* \mathcal{O}(-\mu_2)}{f^* L_1 \otimes \pi^* \mathcal{O}(-\mu_2 - m)}$, then there is an induced homomorphism $\eta : f^* L_2 \otimes \pi^* \mathcal{O}(1 - \mu_2 - m) \rightarrow E'$. By the snake lemma, η is injective and E' fits exact sequences

$$0 \rightarrow f^* L_1 \otimes \pi^* \mathcal{O}(-\mu_2 - m) \rightarrow E \otimes \pi^* \mathcal{O}(-\mu_2) \rightarrow E' \rightarrow 0 \quad (64)$$

and

$$0 \rightarrow f^* L_2 \otimes \pi^* \mathcal{O}(1 - \mu_2 - m) \rightarrow E' \rightarrow f^* L_3 \rightarrow 0. \quad (65)$$

The sequence (65) induces a morphism $q_2 : B \rightarrow \mathbb{P}(H^1(L_3^{-1} \otimes L_2))$ such that $q_2^* \mathcal{O}_{\mathbb{P}(H^1(L_3^{-1} \otimes L_2))}(1)$ is of degree $1 - \mu_2 - m = (\mu_1 - \mu_2) + \frac{1}{2}$. $\Delta(E) = 6$ and (12) imply $\deg E_1 = 0$ and $\mu_1 - \mu_2 = \frac{3}{2}$, which also imply $\deg L_1 = \deg L_2 = 0$ and $\deg L_3 = 1$. Thus $q_2^* \mathcal{O}_{\mathbb{P}(H^1(L_3^{-1} \otimes L_2))}(1)$ is of degree $1 - \mu_2 - m = (\mu_1 - \mu_2) + \frac{1}{2} = 2$. The sequence (64) induces a morphism $\psi : B \rightarrow \mathcal{P}$ such that $\psi^* \mathcal{O}_\mathcal{P}(1)$ is of degree $-\mu_2 - m = 1$.

In Proposition 4.11 when E is semi-stable of degree 0 at generic fiber of $f : X \rightarrow C$, there exists only one point $t_0 \in C$ such that E_{t_0} is not semi-stable and the minimal slop quotient bundle G of rank 2 and $\deg G = -1$. \square

Remark 4.13. *If $\phi : B \rightarrow M$ satisfy condition (1), it is a elliptic curve of split type with minimal degree. If $\phi : B \rightarrow M$ satisfy condition (2), it is a elliptic curve in Proposition 3.6. which implies a elliptic curve of degree 6 may not be a elliptic curve of split type.*

Theorem 4.14. *When $g > 12$ and C is generic, any essential elliptic curve $\phi : B \rightarrow M = SU_C(3, \mathcal{L})$ that passes through the generic points must have $\deg \phi^*(-K_M) \geq 18$.*

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